Application of Pseudo-Wavelets to Optimal Control

by

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Abstract

Wavelets are a new mathematical concept for basis functions that are currently being researched in many areas including data compression, image processing, and numerical methods. The main characteristic that distinguishes wavelets from most other classical basis functions is that relatively few wavelet approximating functions are necessary to capture abrupt changes in a function which occur in a local area. This thesis demonstrates the effectiveness of applying wavelets to numerical methods. In particular, wavelets will be applied to the Measured Equation of Invariance (MEI) Method and the Galerkin Method. The MEI method can be seen as an improvement on the finite difference method and has been shown to be computationally fast. The Galerkin method is a numerical method which can be used to solve equations in optimal control theory. The Galerkin method expands the solution in terms of a linear differential combination of basis functions. In this paper, functions that have wavelet - like properties, called pseudo wavelets, will be used in the Galerkin method. Formulations for the Galerkin method for optimal control problems involving a state vector and a control vector of arbitrary size will be discussed. In addition to this, formulations for the Galerkin method will be provided for a time - delay problem in optimal control. The effectiveness of these pseudo - wavelets will be apparent. In many instances, it has been found that wavelets can be used to decrease computational time and converge to an accurate solution with fewer basis functions.

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Chapter 1: Wavelets

1.1 Introduction

Wavelets are a recent mathematical concept and have been applied with very satisfactory results in many areas. In the past, Fourier Analysis has been applied to the areas of data compression, image processing, and numerical methods. Most basis functions used in Fourier Analysis such as sines, cosines, and Legendre functions have the property that they are entire domain, that is that their value does not damp out as independent variable approaches infinity or negative infinity. This makes these functions well suited for dealing with phenomena that occur continuously and smoothly. However, many things in nature involve abrupt changes and sharp peaks that happen over a small interval of time or space. This will be referred to as localized behavior. Such localized behavior is not well approximated by basis functions that are undamped and exist on the entire interval of interest. Since wavelets posses localized behavior, they approximate functions with abrupt changes well. Thus, wavelets have been used in areas of data compression, signal processing, and numerical methods. In particular, wavelets have been used to solve many different partial differential equations [4], [8], [12].

1.2 Properties of Wavelets

Wavelets are functions that form a basis, but have damped behavior and have most of their value in a small area. Wavelets are functions of the form $\Psi(nx-k)$ which can be produced by translations and dilations of some mother wavelet, $\Psi(x)$. To be a wavelet, this function must satisfy the admissibility condition:

$$\int_{-\infty}^{\infty} \frac{\left|\bar{\psi}(\omega)\right|^2}{|\omega|} d\omega \quad < \quad \infty \tag{1.1}$$

[5]. Here, the carrot is used to denote the Fourier transform. Since this integral is finite, it forces the Fourier transform of Ψ evaluated at $\omega = 0$ to be zero. This means that

$$\int_{-\infty}^{\infty} \psi(t)dt = 0. \tag{1.2}$$

Since the entire family of wavelets is produced from translations and dilations of this mother wavelet, $\psi(t)$, it follows that all of them must have integral zero. Furthermore, since their integral over the entire real axis is finite and well defined, it follows that they must oscillate and damp out. This damping effect causes wavelets to be well suited for applications involving functions with localized behavior.

1.3 Daubechies Wavelets

This section discusses a special class of wavelets that were specially designed to be orthogonal and to approximate higher order curves. Dr. Ingred Daubechies has introduced a mathematical framework for the creation of such wavelets [6]. These conditions make the generated wavelets a powerful approximation tool.

To begin with, the wavelets are generated from a scaling function. This scaling function is required to satisfy special requirements. The first requirement that the scaling function, $\Phi(t)$, must satisfy is the dilation equation:

$$\Phi(t) = c_1\Phi(2t) + c_2\Phi(2t\text{-}2) + ... + c_n\Phi(2t\text{-}n\text{+}1) \tag{1.3}$$

This dilation equation ensures that the scaling function can be written as a linear combination of its smaller dilates. These constants c_i are determined with equations that are given to make the resulting wavelet have desirable properties. These conditions are the normality condition, the orthogonality, and a set of n-2 moment conditions.

Normality Condition:

$$c_1 + c_2 + \ldots + c_n = 2 \qquad (1.4)$$

The normality condition normalizes the size of the scaling function and can be obtained by integrating the dilation equation over (- ,).

Orthogonality Condition:

$$c_1\, c_3 + c_2\, c_4 + c_3\, c_5 + \ldots + c_{n-2}\, c_n = 0 \qquad (1.5)$$

This condition causes a wavelet to be orthogonal to the other wavelets.

n-2 Moment Conditions

$$c_1 - 2^k c_2 + 3^k c_3 - \ldots + n^k c_n = 0 \qquad (1.6)$$

$$k = 0, 1, \ldots, n-2$$

The moment conditions ensure that the scaling function can approximate curves of order n-2 and less.

Once the constants are determined from these equations, the dilation equation is solved to find the scaling function. Once the scaling function is found, the actual wavelet is generated from the scaling function via the following equation:

$$\psi(t) = c_n\, \phi(2t) + c_{n-1}\, \phi(2t-1) + c_{n-2}\, \phi(2t-2) + \ldots + c_1\, \phi(2t-n+1) \qquad (1.7)$$

When two coefficients are used, the resulting wavelet is the D2 wavelet, also called the Haar wavelet. Pictures of the scaling function in this case and the resulting Haar wavelet are shown below.

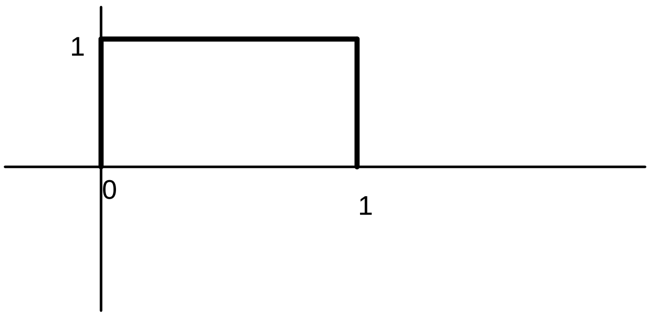

Figure 1.1 The scaling function for the Haar wavelet

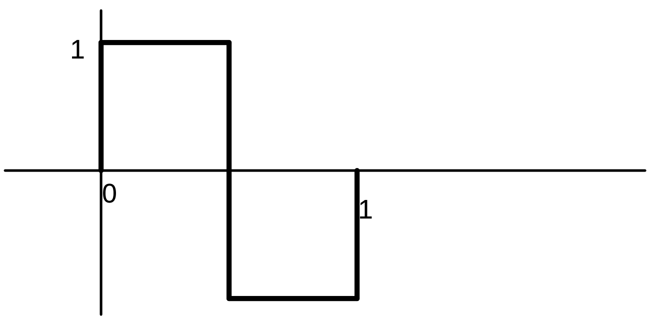

Figure 1.2 The Haar wavelet

It is possible to use more than two coefficients in the scaling equation. When this is done, the resulting wavelet will be able to approximate higher order functions. Many of these resulting wavelets are very irregular in shape, but have great approximating power.

1.4 Conclusion

In this chapter, the basic properties of wavelets and ways that they can be generated has been discussed. Wavelets can be used in many instances to improve efficiency and decrease computational time. This is due to their highly localized behavior.

This highly localized behavior is also useful in a concept called multiresolution. Using the concept of multiresolution, fewer wavelet terms can be used where the solution is not rapidly changing and more terms can be used where the solution is changing more rapidly. This makes the algorithm more efficient. In addition to their use in multiresolution analysis, there are also fast wavelet algorithms for integration and integral transforms.

Chapter 2: The MEI Method

2.1 Introduction

In this chapter, numerical methods for calculating the electric field due to a plane wave that is incident upon a cylinder of arbitrary cross section will be discussed. In general, analytic solutions do not exist for this problem and it is important to have fast and efficient numerical algorithms for solving this problem. The MEI method can be used to write modified finite difference equations which in turn leads to increased computational speed for large problems. Results will be obtained from applying the Haar wavelets to the MEI method.

2.2 Description of the Scattering Problem

The basic problem involves a known electric field, E_{inc} , which strikes an infinite conducting cylinder of known cross section. A current is induced on the cylinder and this in turn produces an electric field, E_{scat}. The resultant field is the sum of the incident and scattered fields.

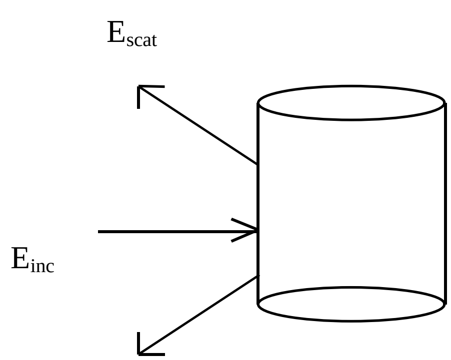

Figure 2.1 The General Scattering Problem

If the metal object can be assumed to be of uniform cross section, then the governing equation in this situation is the Helmholtz equation, which gives the total electric field. The Helmholtz equation is of the form:

$$\nabla^2\Phi + k^2\Phi = 0 \tag{2.1}$$

This equation is subject to boundary conditions, one of which is the free space condition that states that the scattered field must go to zero as one gets infinitely far away from the scatterer. This condition is known as the radiation condition. The other boundary condition is determined at the boundary of the cross section of the conductor. In special instances, such as in the case of a circular cross section, this problem can be solved analytically and there is no need for numerical methods. Most of these ideal cases have already been solved. Practically, for most cross sections like a square, the problem can not be solved analytically, and one must resort to numerical methods.

2.3 Overview of the Finite Difference Method

In the finite difference method, the solution region is divided into a finite set of grid points and derivatives are approximated as finite differences. This typically leads to a linear system of equations that can be solved to find the approximate value of the unknown quantity at the grid points. One of the major shortcomings of the finite difference technique is that it is not very efficient for solving problems where the solution region is open. In these cases where the solution region is open, it is necessary to introduce a fictitious outer boundary with some special conditions. The Absorbing Boundary Condition (ABC), approximates the electric field to be zero on the fictitious outer boundary. If this boundary is chosen close to the area of interest, the results will not be accurate. If this boundary is chosen farther away from the area of interest, the results will become more accurate but computational time will be greatly increased.

2.4 The MEI Method

The MEI Method, developed by Kenneth Mei makes it possible to make accurate calculations while still bringing the fictitious outer boundary closer to the area of interest. Typically, this grid needs only to be about two or three layers. For the purposes of this thesis, this solution region will be assumed to contain an object which is a perfect conducting cylinder of arbitrary cross-sectional shape and size. Near the object, but not on the boundary, is the area of interest where the solution is sought. To solve the problem, it is necessary to find the currents on the surface of the cylinder. After this is found, then the Green’s function for the problem can be used to find the scattered field anywhere.

To begin with, the region is truncated and divided with different grid points, just as in the finite difference method. The differential equation governing the electric field, the Helmholtz Equation, is expressed in the form of a finite difference equation. Since a finite region is used to approximate an infinite region, some conditions for the boundary points are necessary. To accommodate for the fictitious outer boundary, a special set of equations is derived to be applied at the outer boundary.

In general, a finite difference equation will relate a grid point to its neighboring grid points via some finite difference equation that can be expressed as below.

$$\sum_{i=1}^{N} a_i \Phi_i = 0 \tag{2.2}$$

For grid points near the object, the coefficients a_i are fully determined by the differential equation involved. At the truncated grid boundary, the exact values of these coefficients are determined via the MEI method. The coefficients change with the location of the object grid points. The nature of these equations is determined by the geometry of the problem.

Since the object is a perfect electric conducting cylinder, the electric field is normal to the boundary and a current is allowed to exist on the surface of the object. To determine the value of each coefficient, a_i , the surface current density is divided into two parts. J_{error} is the residual part that can not be represented by a set of chosen linearly independent functions called metrons. J_{met} is the part that can be represented by linearly independent metrons, σ_k. This representation is shown below.

$$J_{met} = \sum_{i=1}^{N} c_k \sigma_k \tag{2.2}$$

When a metron is multiplied by the Green's function for the problem and integrated over the boundary of the object, the resulting quantity is the field. This field can be evaluated at the outer edge of the grid. After this is repeated for three or more metrons, one obtains a linear system of equations for the a_i's in equation (2.2) of the outer MEI layer. This system for the a_i's can be solved by any linear equation solver. Then standard finite difference can be used for the entire truncated grid.

2.5 Results

A MATLAB program written by Rafael Pous © 1994 was modified to apply the Haar wavelets as metrons in equation (2.2) to the scattering problem and the results were compared to the case where sinusoidal metrons were used. It was found that the numerical solution obtained using the Haar Wavelets as metrons and sinusoids as metrons were identical. Computer times also were unaffected. Figures 2.2 - 2.7 show the results when the MEI method is applied to a cylinder of square cross section and a cylinder of ellipsoidal cross section.

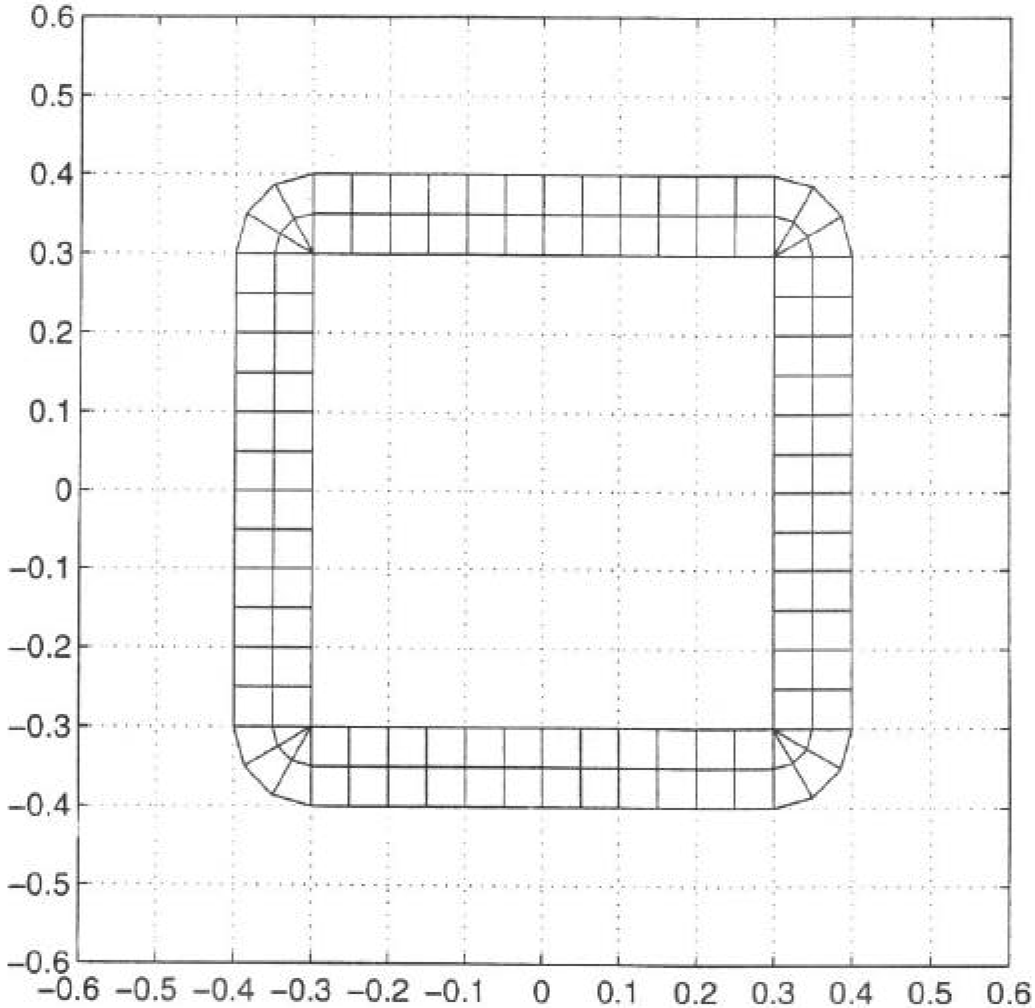

Figure 2.2 The Square Cylinder used in Figures 2.3 and 2.4 with Dimensions Expressed in terms of a Wavelength

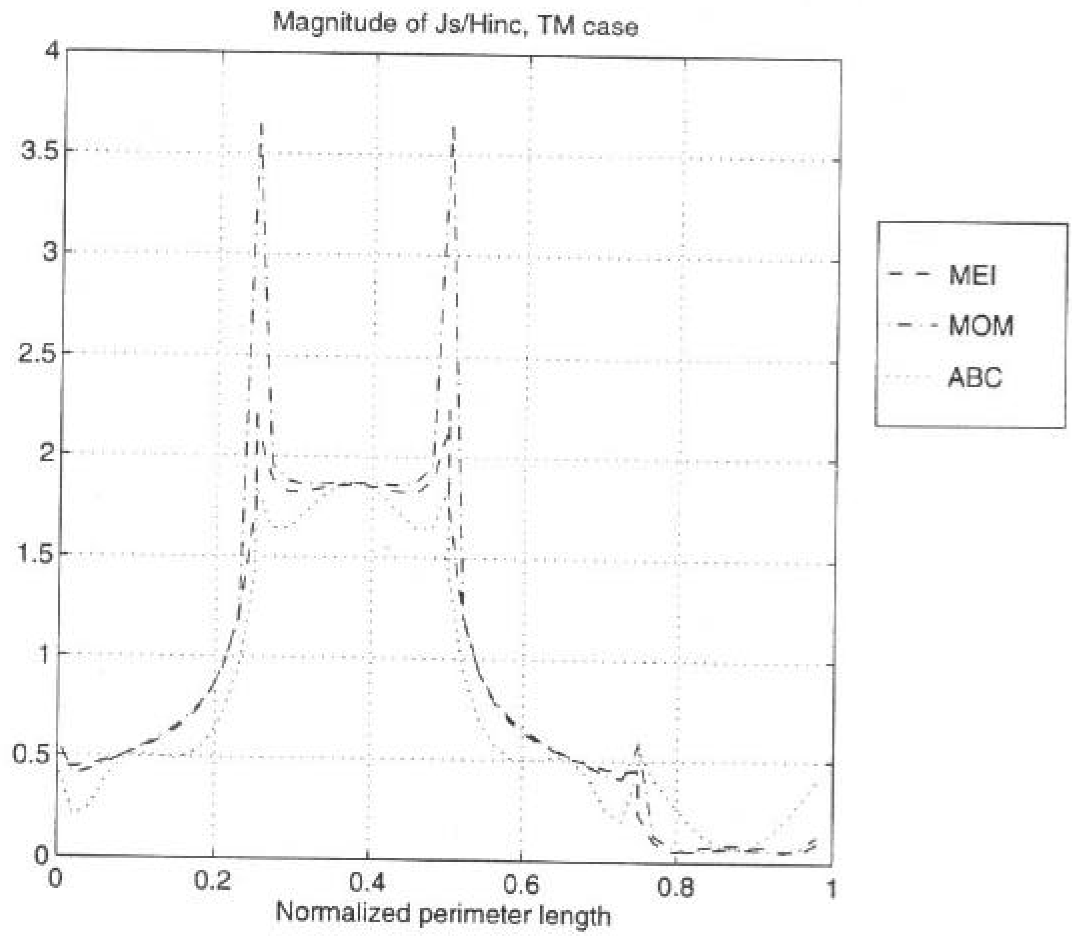

Figure 2.3 Currents Induced on a Square Cylinder with a Three Layer Grid Computed using Five Haar Wavelets as Metrons by an Incident Wave 30 degrees from Normal

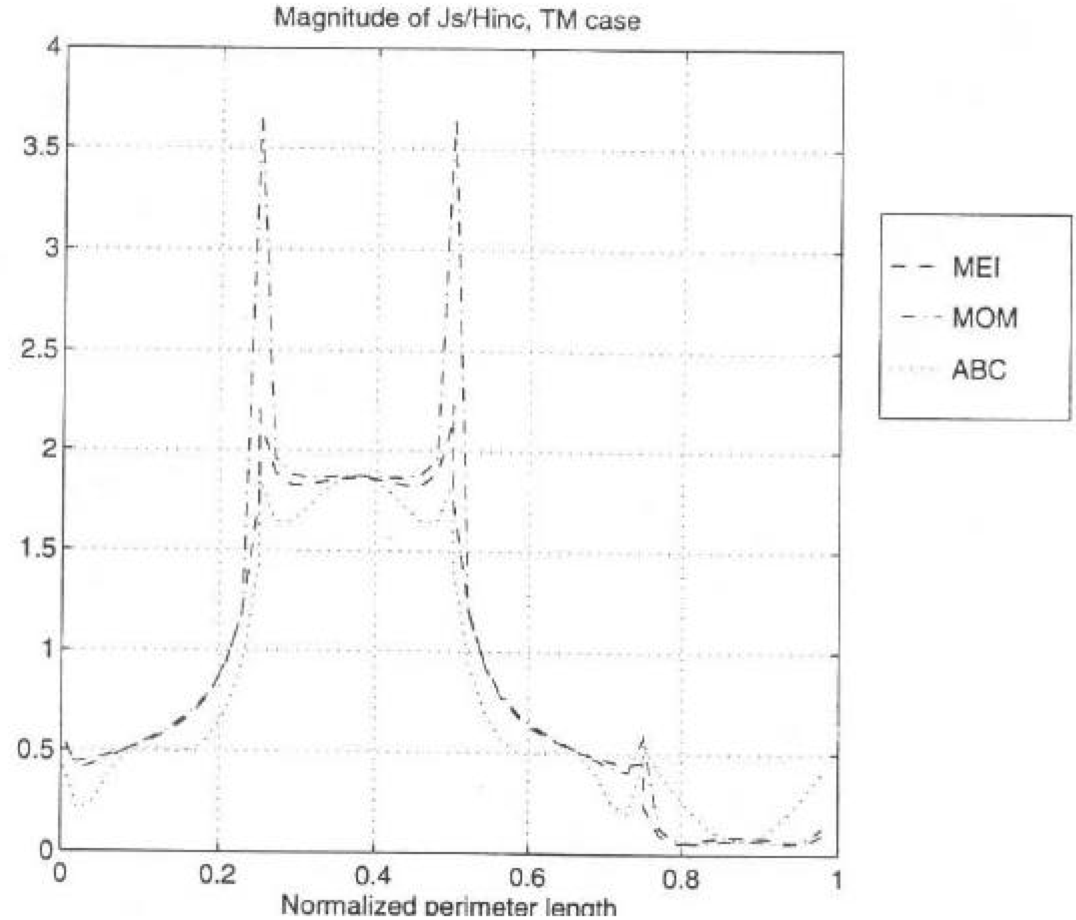

Figure 2.4 Currents Induced on a Square Cylinder with a Three Layer Grid Computed using Five Sinusoidal Functions as Metrons by an Incident Wave 30 degrees from Normal

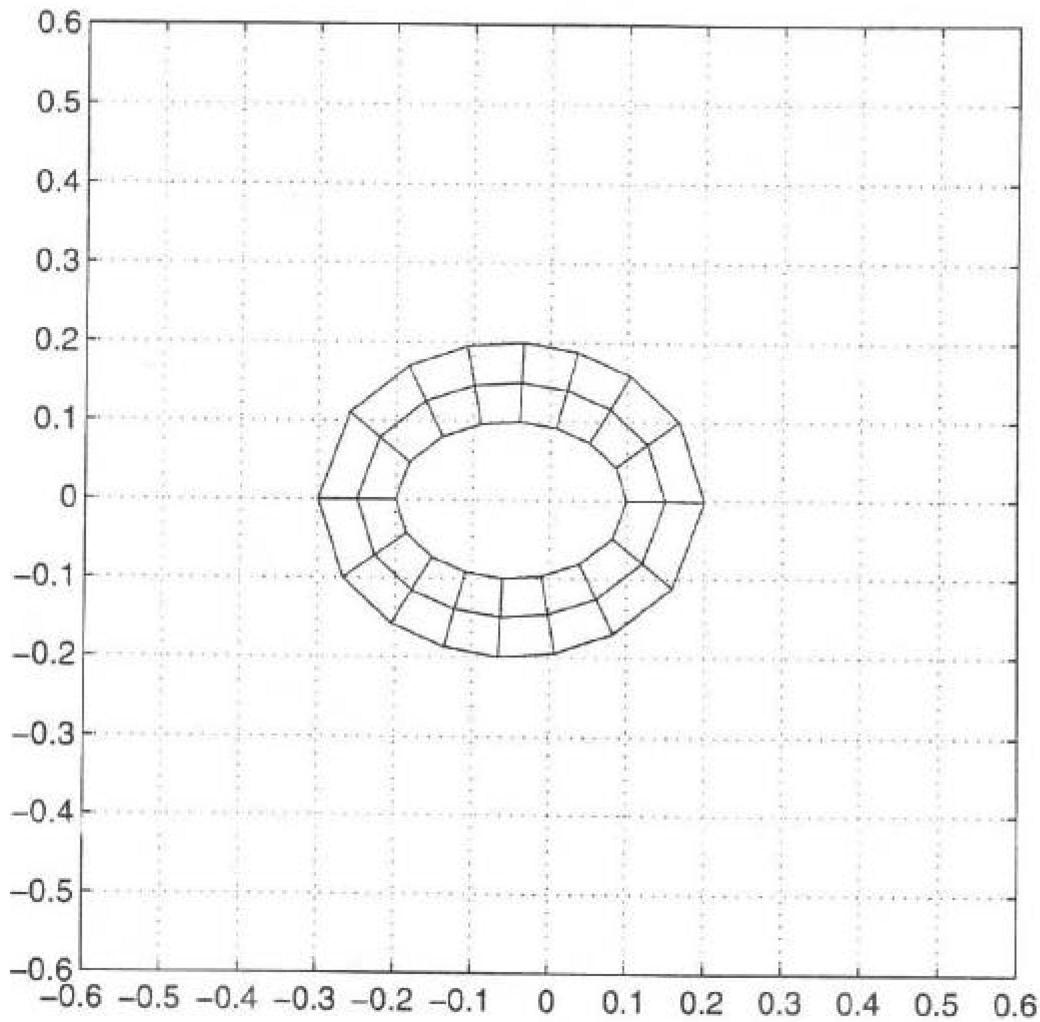

Figure 2.5 The Ellipsoidal Cylinder used in Figures 2.6 and 2.7 with Dimensions Expressed in terms of a Wavelength

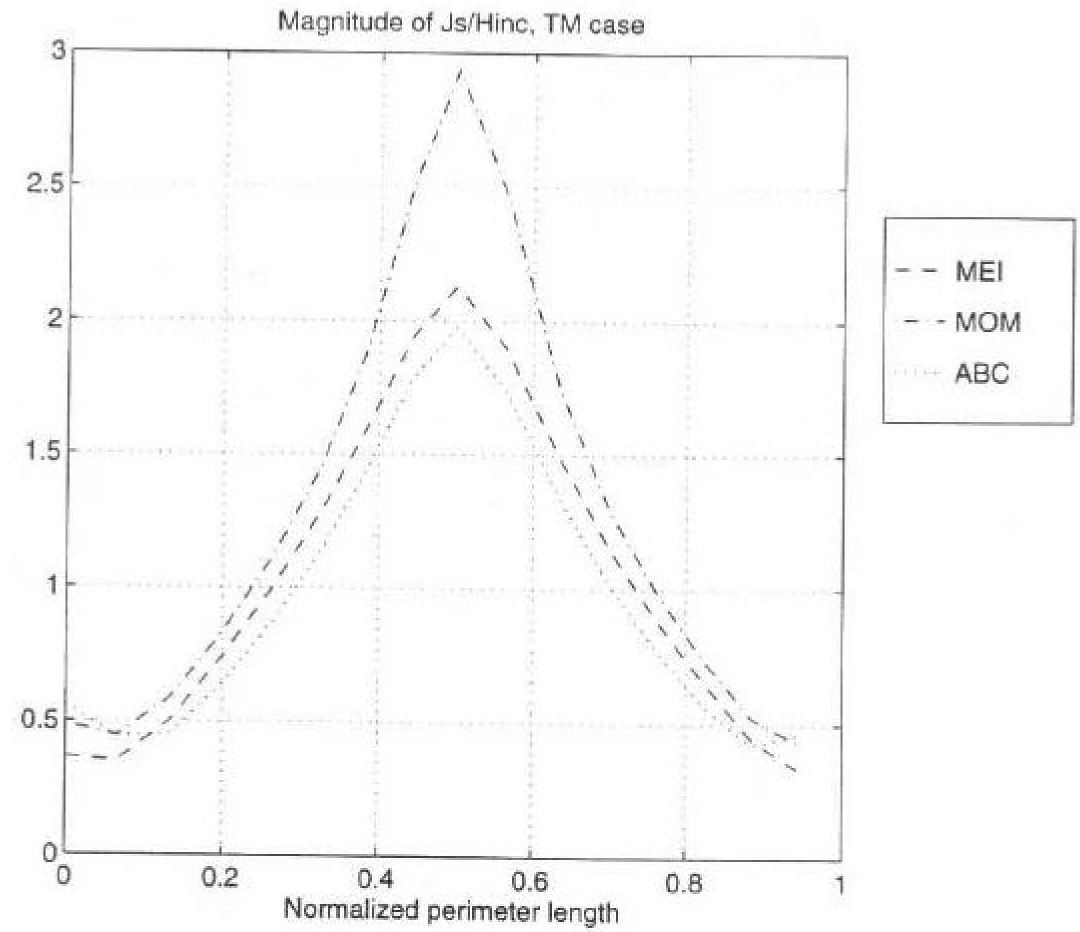

Figure 2.6 Currents Induced on a Ellipsoidal Cylinder with a Three Layer Grid computed using five Haar Wavelets as Metrons by a Wave 30 degrees off Normal Incidence

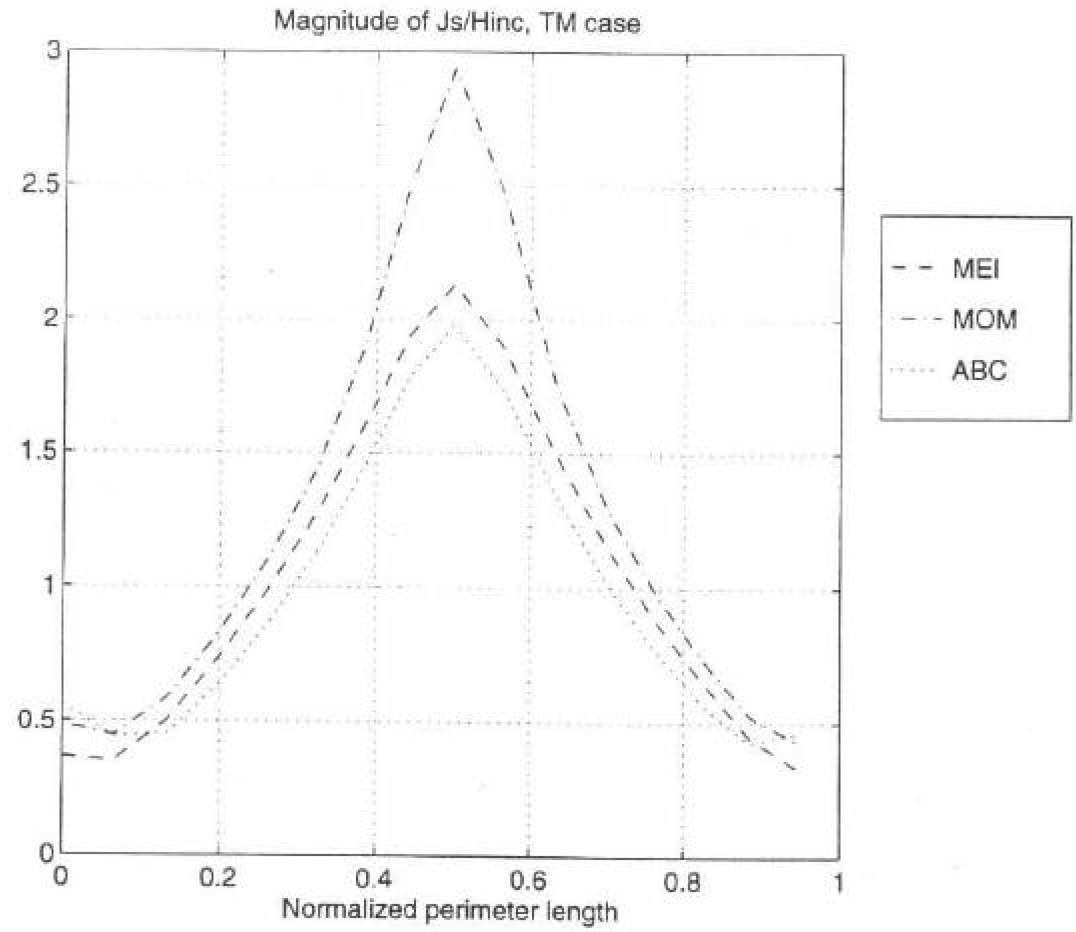

Figure 2.7 Currents Induced on a Ellipsoidal Cylinder with a Three Layer Grid computed using Five Sinusoidal Functions as Metrons by a Wave 30 degrees off Normal Incidence

2.6 Conclusion

The Haar wavelets do give as accurate results as the sine and cosine functions when used as metrons for the MEI Method. In the results obtained in this thesis, the computational times were almost identical. These were all from an existing MATLAB program which was modified to use the Haar wavelets. No effort was made to make the computational times faster. However, there do exist fast algorithms for integration when the Haar wavelets are involved. This integration is necessary when equation (2.2) is multiplied by the Green's Function and integrated. Similar research has been done at Southern Illinois University by Byunge Lee and Dr. Frances Harackiewicz and it was found that the Haar wavelets can be used to significantly decrease the computational time over sinusoidal functions for large problems.

Chapter 3: Pseudo Wavelets

3.1 Introduction

For the results of this thesis in optimal control, a family of basis functions which is produced from translations and dilations of a mother function was used. In this sense, the set of basis functions used is similar to a set wavelets. However, the members in this set of basis functions used do not have integral zero. For this reason, this set of basis functions will can not be considered wavelets and will therefore be referred to as pseudo-wavelets. This chapter discusses the formulation, completeness, and linear independence of these pseudo wavelets.

3.2 Formulation

For the purposes of this thesis, the function $H_n(t)$ is defined as follows:

$$H_n(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < \mathrm{mod}(n+1, L)/L \\ -1 & \mathrm{mod}(n+1, L)/L < t < 1 \\ 0 & t > 1 \end{cases} \quad \text{where } L = \lceil \log_2(n+1) \rceil, \quad n > 1 \tag{3.1a}$$

$$H_1(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{3.1b}$$

Figure 3.1 The Haar Basis

Note that these functions are very similar to the Haar wavelets except for the fact that they exist only on the interval [0,1] and there is a constant function added as the first basis function. These functions technically are not wavelets, but they can be in place of the Haar wavelets in problems where the solution region is a closed interval as opposed to the entire real axis. For this reason, this family of functions will be referred to as the Haar basis (Figure 3.1).

The pseudo-wavelets used in this thesis are created by taking the indefinite integral of these functions. These new pseudo-wavelets will be referred to as "Triangle Functions".

$$\phi_n(t) = \int_0^t H_n(s)ds, \text{ for } n > 2 \tag{3.2}$$

These triangle functions exist only on the interval [0,1] and have the following appearance:

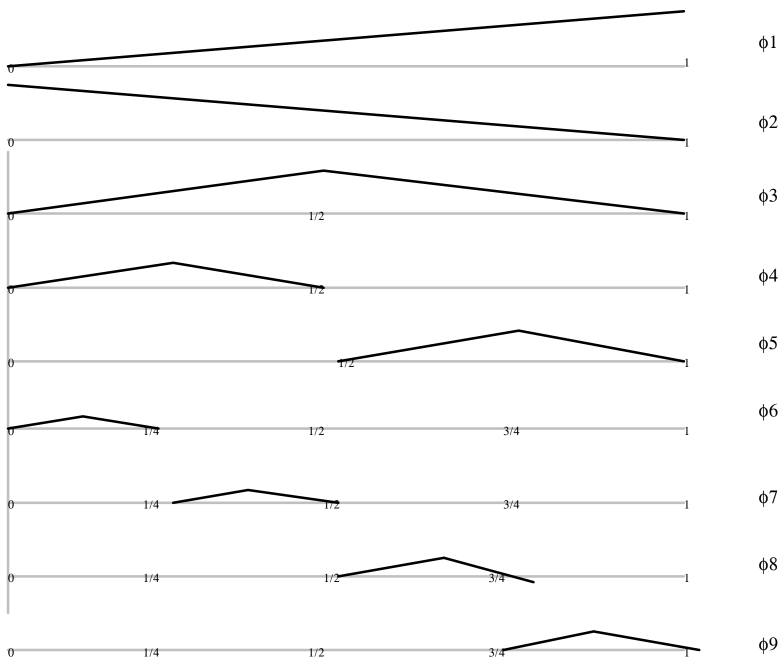

Figure 3.2 The Triangle Functions or Pseudo Wavelets

Note that this term by term integration produces an arbitrary constant, and the integral of the H_1 function is an increasing ramp function. ϕ_1 and ϕ_2 span the same space as these two functions, and will be used for mathematical convenience. These pseudo-wavelets are not wavelets, since they do not have integral zero. However, they still have wavelet-like properties such as localized behavior. Functions very similar to these have been applied to solve minimum energy control problems [10]. These functions have also been used previously to solve Maxwell's equations [13] using a Galerkin Method. The properties of completeness and linear independence of these triangle functions are not obvious.

3.3 Proof of Completeness

The Haar basis can be shown to span the set of all continuous functions on the interval [0,1] by the following argument. $H_1(t)$ can be used to represent the average value of the function in the interval [0,1]. $H_2(t)$ can be added to this in order to represent the average value of the function on the interval [1,0.5] and the interval [0.5,1]. $H_3(t)$, $H_4(t)$, $H_5(t)$, and $H_6(t)$ can be added in linear combination to the previous linear combination of functions to represent the average value of the function on the intervals [0,0.25], [0.25,0.5], [0.5,0.75], [0.75,1]. This process can be continued so that the average value of any function can be represented on arbitrarily small intervals, and completeness follows. It therefore follows that the derivative of an arbitrary function, f(t), can be expanded as follows: $\dot{f}(t) = a_1H_1(t) + a_2H_2(t) + \ldots$. If this function is uniformly continuous, it follows that $f(t) = a_1\phi_1(t) + a_2\phi_2(t) + \ldots$. Now if one considers $\{ H_n(t)\ n=1,2,3,\ldots \}$, it should be apparent that these functions are orthogonal. The set of functions $\{ \phi_n(t)\ n = 1,2,3, \ldots\}$ is not orthogonal, but does form a complete basis for the set of continuous functions on the interval [0,1] and contains elements that are all linearly independent.

3.4 Proof of Linear Independence

The argument for linear independence goes as follows. Consider ϕ_1 and ϕ_2. These functions are linearly independent since at the boundaries (1 and 0) one of these functions is zero while the other function is 1. Note that any function formed by linear combinations of ϕ_1 and ϕ_2 will be differentiable at the point t=1/2. ϕ_3 is not differentiable at the point t=1/2, so it follows that ϕ_1, ϕ_2 , and ϕ_3 are linearly independent. It should be obvious that ϕ_4 and ϕ_5 are independent of each other and not differentiable at the points t=1/4 and t=3/4, respectively. Since any linear combination of ϕ_1 , ϕ_2 , and ϕ_3 would be differentiable at these two points, it follows that all five of these functions are linearly independent. The functions ϕ_6 , ϕ_7 , ϕ_8 , and ϕ_9 are linearly independent of each other and not differentiable at the points 1/8, 3/8, 5/8, and 7/8, respectively. Since any linear combination of ϕ_1 , ϕ_2 , ϕ_3 , ϕ_4 , and ϕ_5 will be differentiable at these points it follows that all nine of these functions are linearly independent. This proof continues like a proof by induction, adding twice as many functions as was added in the previous step and establishing linear independence for the whole set.

3.5 Conclusion

Now that completeness and linear independence of these functions has been established, they will be used in the following chapters to solve problems in optimal control. One very desirable property apparent when these functions are used to represent the solution of some problem, i.e. $x(t) = \sum_{i=1}^{N} a_i \phi_i$, is that a_1 is completely determined by the initial conditions of the solution and a_2 is completely determined by the final conditions of the solution.

Chapter 4: The General Optimal Control Problem

4.1 Introduction

In this chapter, a general control problem and the solution technique presented by Om Agrawal [1] will be presented and extended to accommodate a numerical algorithm involving the pseudo wavelets discussed in chapter three.

4.2 The General Control Problem

The control problem of interest is to find the optimal control, u(t), which minimizes the performance index

$$J = \int_0^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt \tag{4.1}$$

which is subject to the constraints

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.2a}$$

$$x(0) = x_0 \tag{4.2b}$$

where x(t) is an m x 1 state vector, u(t) is an r x 1 control vector, Q(t) is an m x m positive semi-definite matrix, R(t) is an r x r positive definite matrix, and A and B are matrices of sizes m x m and m x r, respectively. If A, B, Q, and R are constant matrices then the problem is said to be time invariant. Otherwise the problem is said to be time variant. The performance index, J, often has some relation to the energy expended. The state vector, x(t), is something that one is interested in controlling. The control variable, u(t), is what is applied to control the state vector. Equations (4.2a) and (4.2b) are physical constraints that are often given by the laws of nature.

One example of an optimal control problem [9] involves finding the optimal way to raise the temperature of a room to a higher temperature in a fixed amount of time, while minimizing the total heat used to do this. In this example, the control variable, u(t), is the rate of heat supplied to the room and the performance index indicates the amount of energy used. The state variable, x(t), represents the temperature of the room above the ambient surrounding temperature, and the constraint equation models the heat flow from the room to the outside. The above physical example discussed above illustrates a possible application of the general problem that is described in this chapter.

The optimal control for (4.1)-(4.2b), u(t), has been shown to be [14]

$$u(t) = R^{-1} B^T p(t) \tag{4.3}$$

Where p(t) is determined by the following equations

$$\dot{x}(t) = Ax + BR^{-1} B^T p(t) \tag{4.4a}$$

$$\dot{p}(t) = Qx - A^T p(t) \tag{4.4b}$$

$$x(0) = x_0 \tag{4.4c}$$

$$p(t_f) = 0 \tag{4.4d}$$

Using a lagrange multiplier technique for the calculus of variations, the following equation is obtained [1].

$$\begin{aligned}&\int_0^{t_f} [\delta x^T(\dot{x} - Ax - BR^{-1}B^T p) + \delta p^T(\dot{p} - Qx + A^T p)]dt \\ &+ \delta[\lambda_1^T(x(0) - x_0)] + \delta[\lambda_2^T p(t_f)] = 0\end{aligned} \tag{4.5}$$

If the pseudo-wavelets discussed in chapter 3 are used as approximating functions, the last two terms of equation (4.5) will be zero and automatically satisfied since these pseudo wavelets have the property that the initial condition is determined by a single function and the terminal condition is determined by a single function. This yields the following equation:

$$\int_0^{t_f} [\delta x^T(\dot{x} - Ax - BR^{-1}B^T p) + \delta p^T(\dot{p} - Qx + A^T p)]dt \tag{4.6}$$

Since δx and δp are arbitrarily small variations, it follows that both of the terms in the above equation must vanish. It therefore follows that

$$\int_0^{t_f} \delta x^T(\dot{x} - Ax - BR^{-1}B^T p)dt \tag{4.7}$$

$$\int_0^{t_f} \delta p^T(\dot{p} - Qx + A^T p)]dt \tag{4.8}$$

4.3 Pseudo Wavelet Expansions for Solution

For a numerical solution, the following expansions for the state and control variable are introduced. Note that this expansion is used since it takes into account the initial conditions and the fact that the set of approximating functions can match the initial conditions and terminal conditions each with a single function. This reduces the number of unknown coefficients to be determined.

$$x(t) = x_0\,\phi_2 + \sum_{\substack{k=1 \\ k \neq 2}}^{N} \phi_k(t)\, c_k$$

(4.9)

$$p(t) = 0\,\phi_1 + \sum_{k=2}^{N} \phi_k\, d_k \qquad (4.10)$$

where $x(t)$, c_k, and d_k are m x 1 column vectors yet to be determined.

For the purposes of this problem the following n -1 dimensional vectors are introduced:

$$\Phi_{(1)}(t) = [\ \phi_1(t)\ \ \phi_3(t)\ \ \ldots\ \ \phi_n(t)\ ]^T \qquad (4.11)$$

$$\Phi_{(2)}(t) = [\ \phi_2(t)\ \ \phi_3(t)\ \ \ldots\ \ \phi_n(t)\ ]^T \qquad (4.12)$$

In addition, the following n -1 x m dimensional matrices are introduced.

$$C = [\ c_1\ \ c_2\ \ \ldots\ \ c_m\ ]^T \qquad (4.13)$$

$$D = [\ d_1\ \ d_2\ \ \ldots\ \ d_n\ ]^T \qquad (4.14)$$

Here, c_i and d_i are n-1 x 1 column vectors and are of the form

$$c_i = [\ c_{2i}\ \ c_{3i}\ \ \ldots\ \ c_{ni}\ ]^T \qquad (4.15)$$

$$d_i = [\ d_{1i}\ \ d_{3i}\ \ \ldots\ \ d_{ni}\ ]^T$$

(4.16)

Using the expansions (4.9) and (4.10), the following results are obtained:

$$x(t) = [\ c_1^T\Phi_{(1)}\ \ c_2^T\Phi_{(1)}\ \ \ldots\ \ c_n^T\Phi_{(1)}\ ]^T + x_0\,\phi_2 \qquad (4.17)$$

$$\delta x^T(t) = [\ \delta c_1^T\Phi_{(1)}\ \delta c_2^T\Phi_{(1)}\ \ \ldots\ \delta c_n^T\Phi_{(1)}\ ]^T \qquad (4.18)$$

$$p(t) = [\ d_1^T\Phi_{(2)}\ \ d_2^T\Phi_{(2)}\ \ \ldots\ \ d_n^T\Phi_{(2)}\ ]^T \qquad (4.19)$$

$$\delta p^T(t) = [\delta d_1^T\Phi_{(2)}\ \delta d_2^T\Phi_{(2)}\ \ \ldots\ \delta d_n^T\Phi_{(2)}\ ]^T \qquad (4.20)$$

These equations (4.9)-(4.20) are all expressions that will be used to obtain a linear system of unknowns. The whole idea is to determine the coefficients used in the expressions (4.7) and (4.8) in order to determine the solution.

4.4 System of Equations Obtained with the Pseudo Wavelets

By substituting these expansions into equations (4.7) and (4.8), expanding out the expressions to give a system of equations, and defining some new terms yields the following results.

$$\sum_{i=1}^{m} \delta c_i^T \left[F_1 c_i + e - \sum_{j=1}^{m} \left(G_{ij} c_j + a_{ij} \alpha_{ij} + H_{ij} d_j \right) \right] = 0 \tag{4.21}$$

$$\sum_{i=1}^{m} \delta d_i^T \left[F_2 d_i + \sum_{j=1}^{m} \left(P_{ij} c_j + q_{ij} \beta_{ij} + N_{ij} d_j \right) \right] = 0 \tag{4.22}$$

This is a linear system of equations which can be solved for the vectors c_i and d_i. These are equations (4.25) and (4.26). For equations (4.21) and (4.22) the following terms were introduced for convenience.

F_1 is an (n-1) x (n-1) matrix defined by

$$F_1 = \int_0^{t_f} \Phi_{(1)} \dot{\Phi}_{(1)}^T \, dt \tag{4.23a}$$

e is an (n-1) x 1 column vector defined by

$$e = \int_0^{t_f} \Phi_{(1)}^T \dot{\phi}_2 \, dt \tag{4.23b}$$

G_{ij} is an (n -1) x (n-1) matrix defined by

$$G_{ij} = \int_0^{t_f} a_{ij}\, \Phi_{(1)}\Phi_{(1)}^T dt \qquad (4.23c)$$

where a_{ij} is the element in the i^{th} row and j^{th} column of the matrix A.

α_{ij} is an (n-1) x 1 column vector defined by

$$\alpha_{ij} = x_{0j}\, \Phi_{(1)}\phi_2 \qquad (4.23d)$$

H_{ij} is an (n-1) x (n-1) matrix defined by

$$H_{ij} = \int_0^{t_f} m_{ij}\, \Phi_{(1)}\Phi_{(2)}^T dt \qquad (4.23e)$$

where m_{ij} is the element in the i^{th} row and j^{th} column of the matrix $BR^{-1}B^T$

For equation (4.22) , the following terms were introduced for convenience.

P_{ij} is an (n-1) x (n-1) matrix defined by

$$P_{ij} = \int_0^{t_f} q_{ij}\, \Phi_2\Phi_1^T dt \qquad (4.24a)$$

where q_{ij} is the element in the i^{th} row and j^{th} column of the matrix Q

β_{ij} is a (n -1) x 1 column vector defined by

$$\beta_{ij} = x_{0j}\, \phi_2\Phi_1 \qquad (4.24b)$$

N_{ij} is an (n-1) x (n-1) matrix defined by

$$N_{ij} = \int_0^{t_f} a_{ji}\Phi_2\Phi_2^T dt \qquad (4.24c)$$

F_2 is an (n-1) x(n-1) matrix defined by

$$F_2 = \int_0^{t_f} \Phi_{(2)}(\Phi_{(2)}{}^T)'dt$$

(4.24d)

Using these definitions, equations (4.21) and (4.22) can be expressed in matrix form. These were solved with a LINSYS routine.

$$\begin{bmatrix} F_1 - G_{11} & -G_{12} & \dots & -G_{1m} \\ -G_{21} & F_1 - G_{22} & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ -G_{m1} & & \dots & F_1 - G_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ . \\ . \\ . \\ \mathbf{c_m} \end{bmatrix} - \begin{bmatrix} H_{11} & H_{21} & \dots & H_{m1} \\ H_{21} & & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ H_{m1} & & \dots & H_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ . \\ . \\ . \\ \mathbf{d_m} \end{bmatrix}$$

$$= \int_0^{t_f} \begin{bmatrix} \phi_1\phi_2{}' - \sum_{j=1}^{m} a_{1j} b_j \\ \phi_1\phi_2{}' - \sum_{j=1}^{m} a_{2j} b_j \\ . \\ . \\ . \\ \phi_1\phi_2{}' - \sum_{j=1}^{m} a_{mj} b_j \end{bmatrix} dt \qquad (4.25)$$

$$\left[\begin{bmatrix} P_{11} & P_{21} & \ldots & P_{m1} \\ P_{21} & & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ P_{m1} & & \ldots & P_{mm} \end{bmatrix}\right]\begin{bmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ . \\ . \\ . \\ \mathbf{c_m} \end{bmatrix} - \left[\begin{bmatrix} F_2 - N_{11} & -N_{12} & \ldots & -N_{1m} \\ -N_{21} & F_2 - N_{22} & & . \\ . & & & . \\ . & & & . \\ . & & & . \\ -N_{m1} & & \ldots & F_2 - N_{mm} \end{bmatrix}\right]\begin{bmatrix} \mathbf{d_1} \\ \mathbf{d_2} \\ . \\ . \\ . \\ \mathbf{d_m} \end{bmatrix}$$

$$= \int_0^{t_f} \begin{bmatrix} -\sum_{j=1}^{m} q_{1j}\beta_j \\ -\sum_{j=1}^{m} q_{2j}\beta_j \\ . \\ . \\ . \\ -\sum_{j=1}^{m} q_{mj}\beta_j \end{bmatrix} dt \qquad (4.26)$$

It should be noted that the elements inside these matrices and vectors are matrices and vectors themselves. The matrices on the left hand side of equation (4.25) and (4.26) are of size m(n-1) x m(n-1). The column vector is of size m(n-1).Equation (4.25) and (4.26) can be combined into a single matrix equation of the form- Ax = B. This is a linear set of constant coefficient equations involving 2m(n-1) equations and 2m(n-1) unknowns and can be solved by standard solution methods.

4.5 Conclusion

This section has shown a general formulation an optimal control problem for state and control vectors of arbitrary size. It also has included a complete description of how the Galerkin method can be used to solve such a problem. This method will be applied in the following chapters for the time-invariant and time-variant problems.

Chapter 5: The Time-Invariant Problem

5.1 Introduction

In the previous chapter, a general formulation for applying wavelets to control problems was discussed. A problem is said to be time-invariant if the matrices A, B, Q, and R are constant. In this chapter, a specific problem will be considered and the method will be applied. When the forcing function is zero, the solution can be compared to previously published analytical solutions [1], [7].

For the second part of the chapter, a rectangular pulse forcing function will be considered and the results will be compared to an analytical derived solution.

In both cases, the purpose is not to find the solution, but rather to demonstrate the effectiveness of using these new pseudo-wavelets. This solution was obtained using a FORTRAN program which used the methods outlined in this chapter.

5.2 Description of the Time-Invariant Problem

The specific problem that will be considered in this chapter is stated below.

Minimize
$$J = \int_0^1 [x^2(t) + u^2(t)]\,dt \tag{5.1}$$

subject to:

$$\dot{x}(t) = -x(t) + u(t) \tag{5.2a}$$

$$x(0) = 1 \tag{5.2b}$$

$$u(1) = 0. \tag{5.2c}$$

5.3 Solution Method using Pseudo-Wavelets

It is assumed that both the state variable x(t) and the control variable u(t) can be expanded in terms of the triangle functions, $\phi_n(t)$, discussed chapter three. The solution method used will be the one presented in chapter 4.

$$x(t) = \sum_{k=1}^{N} a_k f_k(t) \tag{5.3}$$

$$u(t) = \sum_{k=1}^{N} b_k f_k \tag{5.4}$$

Note that the initial value x is known and completely determined by the function ϕ_2 . Furthermore, the final value of u is known and is completely determined by ϕ_1 . From this information, the coefficients $a_2 = 1$ and $b_1 = 0$. Using equations (4.11) and (4.12) with the included definitions leads one to the following result:

$$\int_0^1 \Phi_{(1)}(\dot{\Phi}_{(1)}^T + \Phi_{(1)}{}^T)dt * C - \int_0^1 \Phi_{(1)}\Phi_{(2)}{}^T dt * D = \int_0^1 (\Phi_{(1)} - \phi_2 \Phi_{(2)}^T)dt$$

$$-\int_0^1 \Phi_{(2)}\Phi_{(1)}^T dt * C + \int_0^1 \Phi_{(2)}(\dot{\Phi}_{(2)}^T - \Phi_{(2)}^T)dt * D = \int_0^1 \phi_2 \Phi_{(2)}^T dt \tag{5.5}$$

Now if we let X=[C D] and

$$Z = \begin{bmatrix} \int_0^1 \Phi_{(1)}(\dot{\Phi}_{(1)}^T + \Phi_{(1)}{}^T)dt & -\int_0^1 \Phi_{(1)}\Phi_{(2)}{}^T dt \\ -\int_0^1 \Phi_{(2)}\Phi_{(1)}^T dt & \int_0^1 \Phi_{(2)}(\dot{\Phi}_{(2)}^T - \Phi_{(2)}^T)dt \end{bmatrix}$$

$$Y = \begin{bmatrix} \int_0^1 (\Phi_{(1)} - \phi_2 \Phi_{(2)}^T \;\; dt \\ \int_0^1 \phi_2 \quad {}_{(2)} dt \end{bmatrix} \tag{5.7}$$

2n-2 unknowns. This is a sparse system in the sense that many of the elements in the problem matrix are zero.

2 matrices,

$\int_0 \Phi\Phi^T dt$ and $\int_0^1 \Phi\dot{\Phi}^T dt$.

$\Phi(t = [\ \phi_1 \quad \phi_2 \quad \dots \quad \phi_n]$

Once these two matrices are computed, then $\int_0^1 \Phi_i \Phi^T{}_j dt$ and $\int_0^1 \Phi_i \dot{\Phi}_j^T dt$

i, j = 1,2 are just submatrices of these with the $(3-i)^{th}$ row and $(3-j)^{th}$ column excluded.

The first of these matrices is symmetric with many zero entries and all of the entries between 0 and 1/3. This symmetry makes it necessary to compute only half of the elements. There is a further symmetry within this symmetry which makes it necessary to compute only half of these remaining elements. All elements can be computed analytically.

$$\int_0^1 \Phi\Phi^T dt = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \cdot\cdot\frac{1}{64} & \frac{3}{64}\cdot \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{8} & \cdot\frac{3}{64} & \frac{1}{64} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{12} & \frac{1}{64} & \frac{1}{64} \\ \frac{1}{64} & \frac{3}{64} & \frac{1}{64} & \frac{1}{96} & 0 \\ \frac{3}{64} & \frac{1}{64} & \frac{1}{64} & 0 & \frac{1}{96} \end{bmatrix} \tag{5.8}$$

and it will also have zero entries in exactly the same places as the first matrix (disregarding the diagonal). These entries can all be calculated analytically and have a definite and

$$\int_0^1 \Phi(\Phi')^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \cdots \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} & \\ -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{8} & \frac{1}{8} & \\ -\frac{1}{16} & \frac{1}{16} & \frac{1}{16} & .\,0 & 0 & \\ -\frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & .\,0 & 0 & \\ \cdots & & & & & \end{bmatrix} \tag{5.9}$$

5.4 Results for Time-Invariant Problem without the Forcing Function

After the values for the coefficients for c_n and d_n are solved for, they are used to construct the solution for x(t) and u(t). These values are shown in Table 5.1 and Table 5.2 , respectively.

Table 5.1 State Variable, x(t), for the time-invariant problem

Time	Analytical [14]	5 Wavelets	17 Wavelets	513 Wavelets
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.870972	0.882969	0.871761	0.870973
0.2	0.759393	0.765937	0.759771	0.759394
0.3	0.663028	0.668006	0.663362	0.663028
0.4	0.579944	0.589177	0.580491	0.579945
0.5	0.508479	0.510346	0.508600	0.508479
0.6	0.447201	0.454122	0.447663	0.447201
0.7	0.394881	0.397898	0.395063	0.394881
0.8	0.350473	0.352994	0.350634	0.350473
0.9	0.313085	0.319411.	0.313457	0.313085
1.0	0.281970	0.285828	0.282217	0.281970

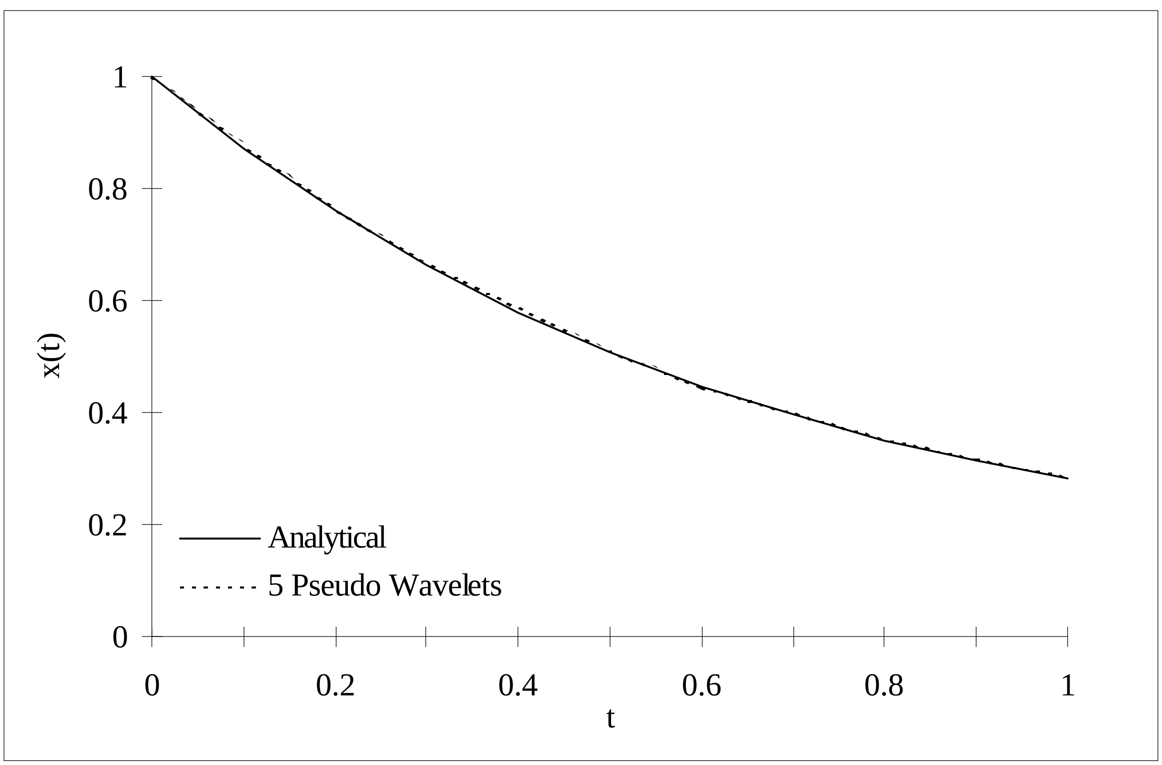

Figure 5.1 State Variable, x(t), for the time-invariant Problem

Table 5.2 Control Variable, u(t), for the time-invariant problem

Time	Analytical [14]	5 Wavelets	17 Wavelets	65 Wavelets
0.0	- 0.385819	- 0.388386	- 0.385983	- 0.385819
0.1	- 0.328060	- 0.332304	- 0.328328	- 0.328060
0.2	- 0.276874	- 0.276222	- 0.276805	- 0.276874
0.3	- 0.231234	- 0.229485	- 0.231142	- 0.231234
0.4	- 0.190227	- 0.192093	- 0.190342	- 0.190227
0.5	- 0.153031	- 0.154701	- 0.153138	- 0.153031
0.6	- 0.118900	- 0.119241	- 0.118924	- 0.118900
0.7	- 0.087151	- 0.083781	- 0.086922	- 0.087151
0.8	- 0.057149	- 0.052841	- 0.056889	- 0.057148
0.9	- 0.028291	- 0.026420	- 0.028171	- 0.028291
1.0	0.000000	- 0.000000	0.000000	0.000000

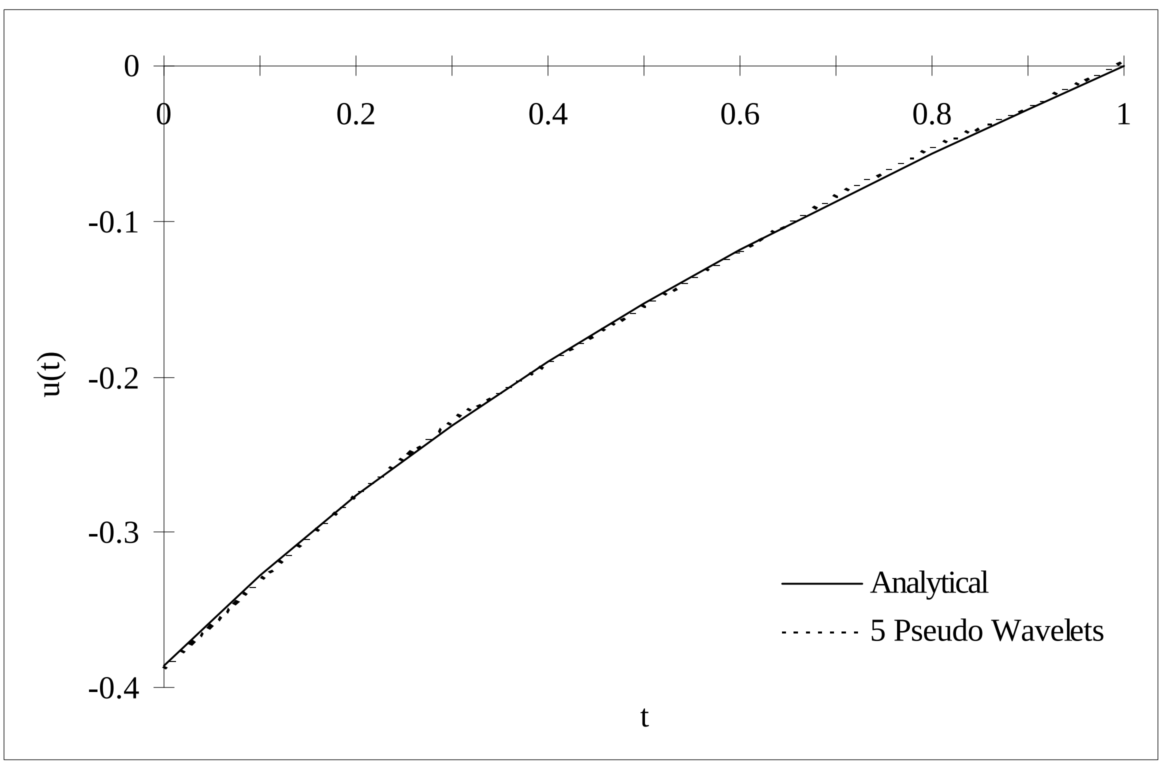

Figure 5.2 Control Variable , u(t), for the time-invariant problem

5.5 The Time-Invariant Problem with a Step Forcing Function

For the second part, a forcing function was added to the dynamic constraints of the problem. This problem can be stated as follows.

$$\text{Minimize} \qquad J = \int_0^1 [\dot{x}^2(t) + \dot{u}^2(t)]dt \tag{5.10}$$

subject to:

$$\dot{x}(t) = -x(t) + u(t) + F(t) \tag{5.11}$$

$$x(0) = 1 \tag{5.12}$$

$$u(1) = 0. \tag{5.13}$$

In equation (5.10), F(t) is an arbitrary rectangular pulse function which has constant value of h on the interval [a,b]. Since no published solutions exist for this problem, the analytical solution will be derived for a pulse function which has height 1 and exists in the interval [0.5,1].

5.6 Derivation of Analytical Solution for the Forcing Function Problem

By applying the standard Euler Lagrange equations one is led to the following equation.

$$\ddot{x}(t) - 2x(t) = \dot{F}(t) - F(t)$$

For this problem, $\dot{F}(t)$, is the difference of two Dirac delta functions which exist where the rectangular forcing pulse, F(t), changes values. Continuing this solution and solving with Laplace transform techniques, one is led to the following solution. Note that although analytical methods were used to obtain this solution, some of the constants were approximated since it was impractical to write down their symbolic value. This is the analytical solution for a rectangular pulse of height 1 which starts at t = 0.5 and ends at t = 1.0.

$$x(t) = -0.00295023 \exp(\sqrt{2}\, t) + 1.00295023 \exp(-\sqrt{2}\, t) +$$

$$s(t-\frac{1}{2})[\frac{1}{2} - \frac{\sqrt{2}+1}{4} \exp(-\sqrt{2}\,(t-\frac{1}{2})) + \frac{\sqrt{2}-1}{4} \exp(\sqrt{2}(t-\frac{1}{2})\}) \quad (5.14)$$

$$u(t) = -0.00712249 \exp(\sqrt{2}\, t) + -0.41543559 \exp(-\sqrt{2}\, t) +$$

$$s(t-\frac{1}{2})[-\frac{1}{2} + \frac{3}{4} \exp(-\sqrt{2}\,(t-\frac{1}{2}) + \frac{1}{4} \exp(\sqrt{2}(t-\frac{1}{2}))]$$

(5.15)

Here, s(t) represents the unit step function.

5.7 Numerical Results for the Forcing Function Problem

The results in Tables 5.3 and 5.4 were obtained using a function which was zero in the interval [0,0.5] and 1 in the interval [0.5,1]. These results are similar to those obtained without the forcing function in the interval [0,0.5] and are noticeably different in the interval [0.5,1].

Table 5.3 State Variable, x(t), for the time-invariant problem with step forcing function

Time	Analytical	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	1.00000	1.00000	1.00000	1.00000	1.00000
0.1	0.86729	0.87663	0.86772	0.86792	0.86732
0.2	0.75195	0.75326	0.75429	0.75200	0.75195
0.3	0.65167	0.65195	0.65366	0.65170	0.65167
0.4	0.56445	0.57267	0.56441	0.56492	0.56448
0.5	0.48854	0.49340	0.48979	0.48855	0.48856
0.6	0.51774	0.52146	0.51836	0.51798	0.51776
0.7	0.54730	0.54951	0.54824	0.54745	0.54731
0.8	0.57780	0.58033	0.57879	0.57795	0.57781
0.9	0.60986	0.61391	0.61049	0.61012	0.60988
1.0	0.64412	0.64750	0.64484	0.64434	0.64414

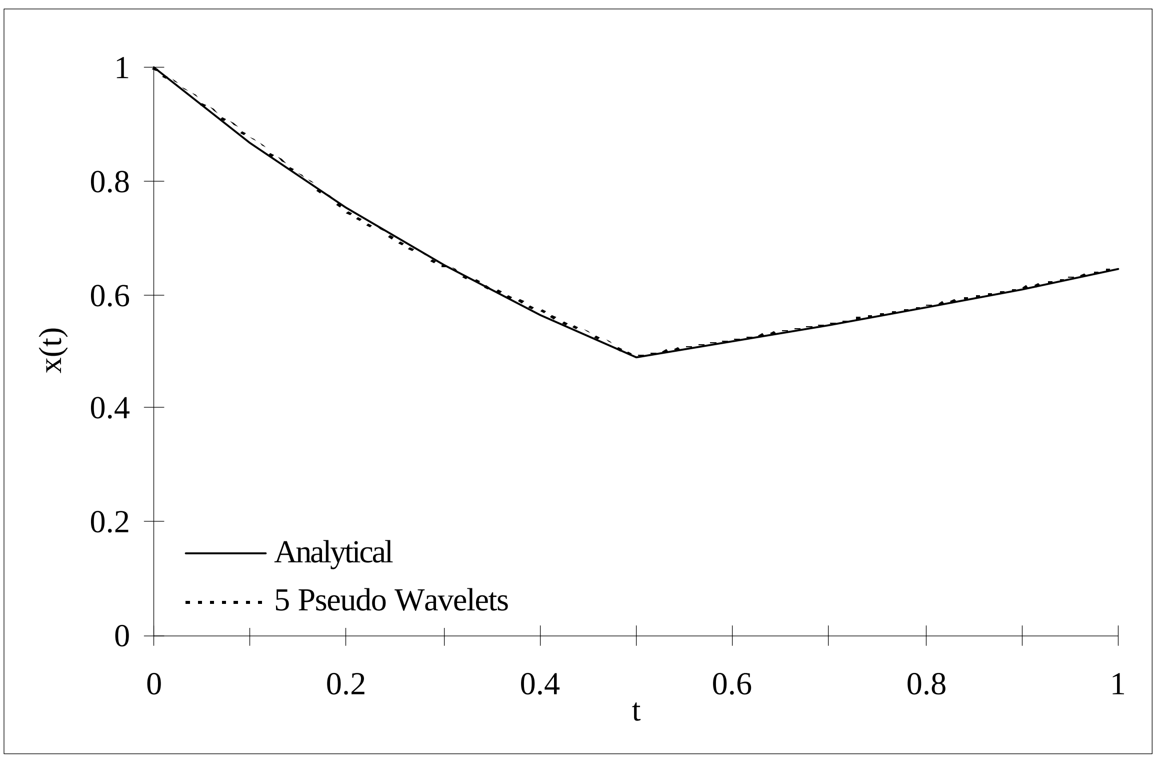

Figure 5.3 State Variable, x(t), for time-invariant problem with step forcing function

Table 5.4 Control Variable, u(t), for the time-invariant problem with step forcing function

Time	Analytical	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	-0.42256	-0.42329	-0.42275	-0.42261	-0.42256
0.1	-0.36885	-0.37160	-0.36826	-0.36904	-0.36886
0.2	-0.32254	-0.31990	-0.32311	-0.32235	-0.32253
0.3	-0.28269	-0.27923	-0.28305	-0.28247	-0.28267
0.4	-0.24850	-0.24958	-0.24750	-0.24855	-0.24850
0.5	-0.21928	-0.21994	-0.21945	-0.21933	-0.21929
0.6	-0.18946	-0.18555	-0.18903	-0.18923	-0.18944
0.7	-0.15341	-0.15116	-0.15231	-0.15326	-0.15340
0.8	-0.11042	-0.10718	-0.10907	-0.11023	-0.11040
0.9	-0.05962	-0.05359	-0.05866	-0.59227	-0.05959
1.0	0.00000	0.00000	0.00000	0.00000	0.00000

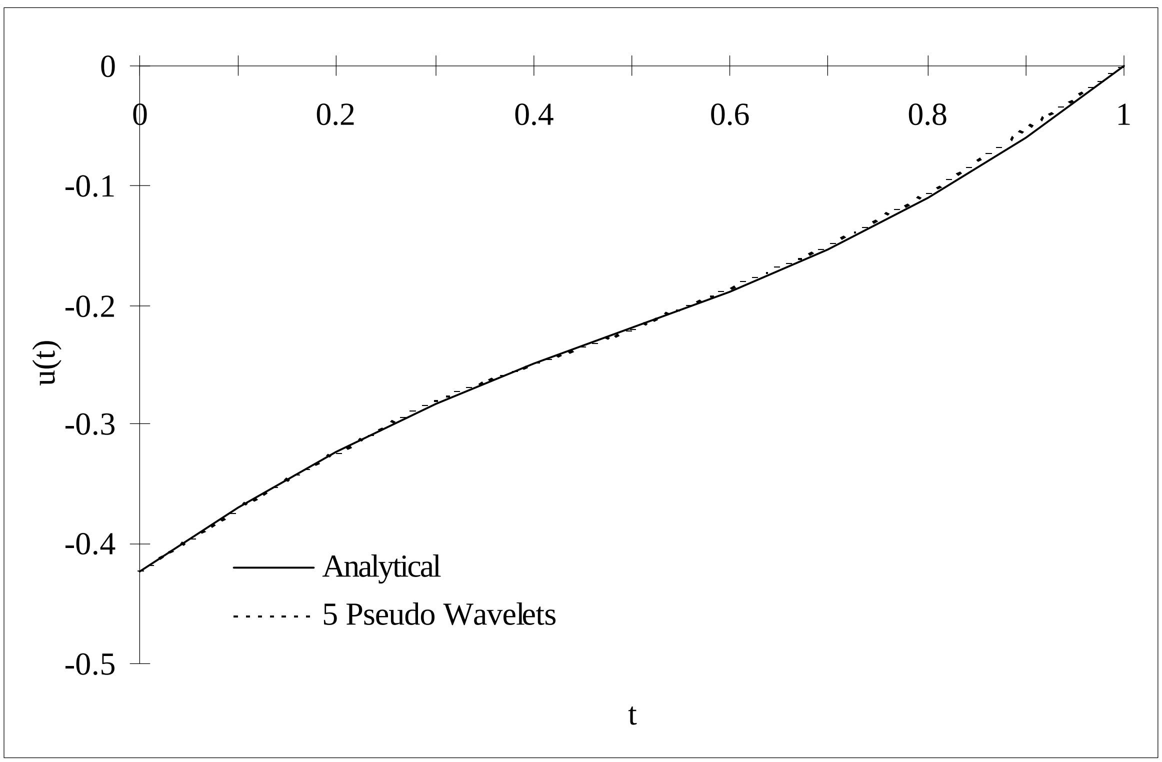

Figure 5.4 Control Variable, u(t), for the time-invariant problem with step forcing function

5.8 Numerical Results for a Rectangular Pulse Forcing Function

After the rectangular step function was considered, $F(t)$ was chosen to be a rectangular pulse which is of amplitude 1 in the interval [0.5,0.75] and 0 elsewhere. This can be used to represent a shock to the system. If this rectangular pulse is chosen to be very narrow and of very high amplitude, then it can be used to represent a delta function.

Table 5.5 State Variable, x(t), for the time-invariant problem with rectangular pulse forcing function

Time	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	1.00000	1.00000	1.00000	1.00000
0.1	0.87294	0.86940	0.86890	0.86822
0.2	0.74587	0.75636	0.75398	0.75377
0.3	0.64609	0.65664	0.65462	0.65444
0.4	0.57359	0.56880	0.56872	0.56824
0.5	0.50109	0.49422	0.49358	0.49339
0.6	0.52433	0.52497	0.52400	0.52376
0.7	0.54757	0.55556	0.55490	0.55462
0.8	0.53358	0.53946	0.53783	0.53776
0.9	0.48234	0.48064	0.48080	0.48041
1.0	0.43111	0.43358	0.43288	0.43266

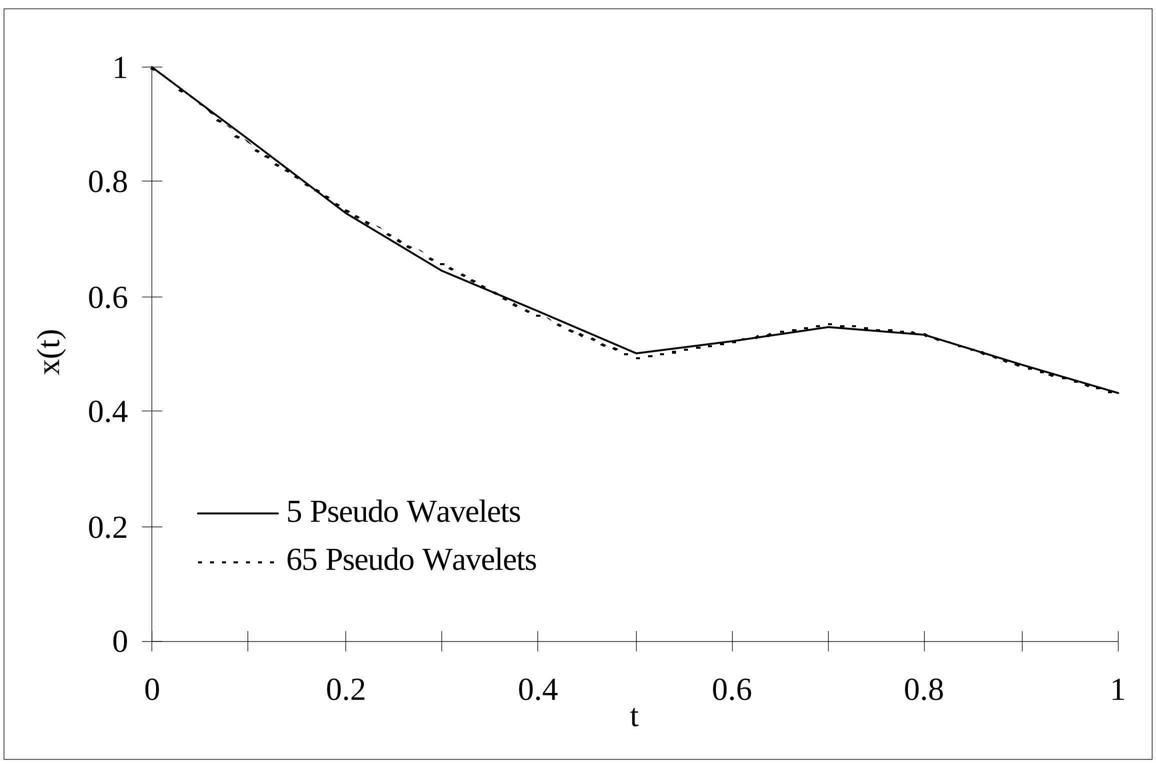

Figure 5.5 State Variable, x(t), for the time-invariant problem with rectangular pulse forcing function

Table 5.6 Control Variable, u(t), for the time-invariant problem with rectangular pulse forcing function

Time	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	-0.41434	-0.41410	-0.41377	-0.41366
0.1	-0.36200	-0.35864	-0.35920	-0.35898
0.2	-0.30966	-0.31222	-0.31134	-0.31146
0.3	-0.26843	-0.27076	-0.27006	-0.27021
0.4	-0.23831	-0.23361	-0.23447	-0.23438
0.5	-0.20818	-0.20350	-0.20330	-0.20323
0.6	-0.17117	-0.17098	-0.17094	-0.17113
0.7	-0.13416	-0.13149	-0.13238	-0.13247
0.8	-0.09253	-0.08730	-0.08736	-0.08767
0.9	-0.04626	-0.04192	-0.04326	-0.04340
1.0	0.00000	0.00000	0.00000	0.00000

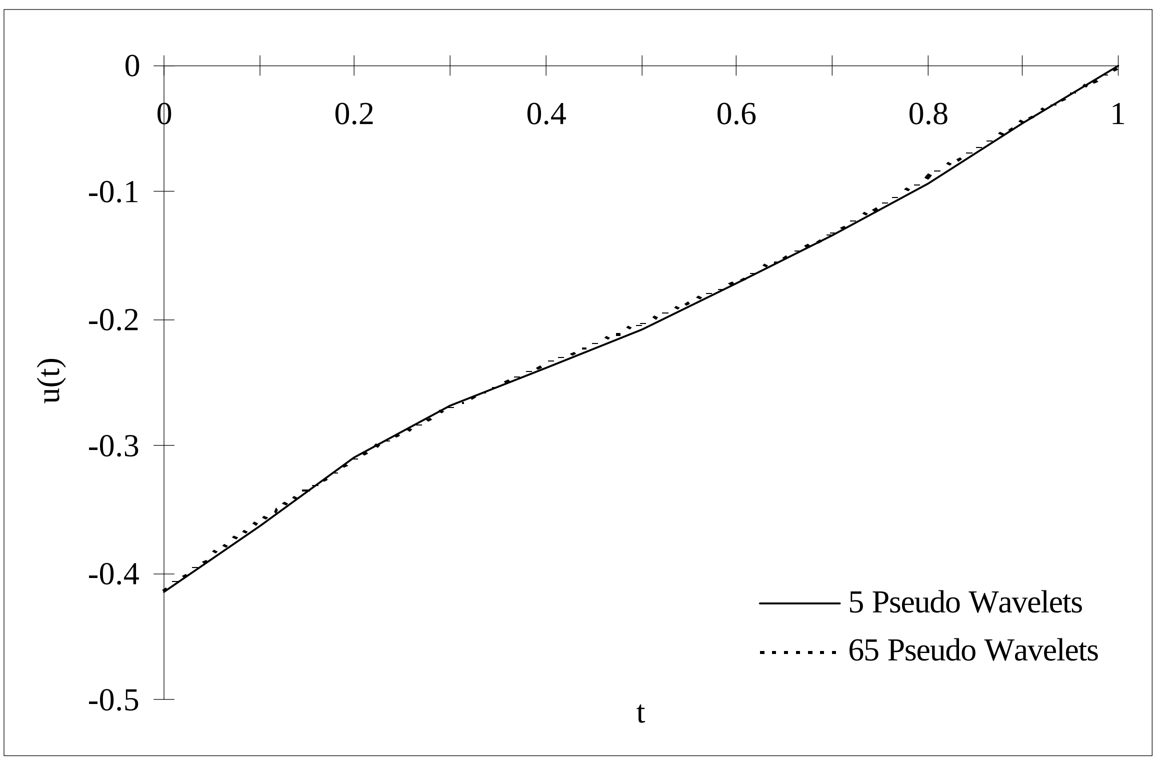

Figure 5.6 Control Variable, u(t), for the time-invariant problem with rectangular pulse forcing function

5.9 Conclusion

In this chapter, pseudo wavelets were used to solve the time-invariant problem. Many published papers demonstrate the use of various orthogonal polynomials and basis functions on this problem [1], [14]. These functions all have the property that they are not designed to capture localized behavior. For problems where there can be erratic changes in the function value, such as when the rectangular pulse forcing function was added, these pseudo-wavelets would be expected to find a more accurate solution. In addition to this, almost all of the basis functions used in the past do not have the property that the initial and final value of the state and control variable are uniquely determined by single basis functions. This reduces the size of the linear system of equations that has to be solved.

Chapter 6: The Time-Variant Problem

6.1 Introduction

For the time-variant problem, a method which identical to the one in the previous chapter will be applied. When the forcing function is zero, the solution can be compared to the general formulation for applying wavelets to control problems that was discussed. in chapter four. A problem is said to be time-invariant if the one of the matrices A, B, Q, or R are time dependent. In this chapter, a specific problem will be considered and compared to previously published solutions.

In both cases, the purpose is not to find the solution, but rather to demonstrate the effectiveness of using these new pseudo-wavelets.

6.2 Description of the Time-Variant Problem

Now a sample time-variant control problem could be as follows:

Minimize $$J = \int_0^1 [\dot{x}^2(t) + \dot{u}^2(t)]dt \tag{6.1}$$

subject to:

$$\dot{x}(t) = tx(t) + u(t) \tag{6.2a}$$

$$x(0) = 1 \tag{6.2b}$$

$$u(1) = 0. \tag{6.2c}$$

Using equations (4.11) and (4.12) and the expansions used in (5.3) same and (5.4) yields the following system of equations

$$\int_0^1 \Phi_{(1)}(\dot{\Phi}^T_{(1)} - t\Phi_{(1)}{}^T)dt * C - \int_0^1 \Phi_{(1)}\Phi_{(2)}{}^T dt * D = \int_0^1 (\Phi_{(1)} + t\phi_2\Phi^T_{(2)})dt$$

$$-\int_0^1 \Phi_{(2)}\Phi^T_{(1)}dt * C + \int_0^1 \Phi_{(2)}(\dot{\Phi}^T_{(2)} + t\Phi^T_{(2)})dt * D = \int_0^1 \phi_2\Phi^T_{(2)}dt \tag{6.3}$$

Now if we let X=[C D] and

$$Z = \begin{bmatrix} \int_0^1 \Phi_{(1)}(\dot{\Phi}_{(1)}^T - t\Phi_{(1)}^T)dt & -\int_0^1 \Phi_{(1)}\Phi_{(2)}^T dt \\ -\int_0^1 \Phi_{(2)}\Phi_{(1)}^T dt & \int_0^1 \Phi_{(2)}(\dot{\Phi}_{(2)}^T + t\Phi_{(2)}^T)dt \end{bmatrix} \qquad (6.4)$$

$$Y = \begin{bmatrix} \int_0^1 (\Phi_{(1)} + t\,\phi_2 \Phi_{(2)}^T)dt \\ \int_0^1 \phi_2 \Phi_{(2)}^T dt \end{bmatrix} \qquad (6.5)$$

then this system can be expressed in the form ZX = Y. This set of equations is similar in nature to the one derived in the previous chapter. This is also a sparse system.

This gives the need to compute 3 matrices, $\int_0^1 \Phi\Phi^T dt$, $\int_0^1 \Phi\dot{\Phi}^T dt$, and $\int_0^1 t\Phi\Phi^T dt$.

The first two of these matrices have already been calculated for the time-invariant problem. The third matrix is symmetric and can be calculated analytically.

This matrix is symmetric and has the zero entries at exactly the same places as the $\int_0^1 \Phi\Phi^T dt$ matrix. After the system of equations is solved to get the coefficients for a_n and b_n , they are used to construct the solution for x(t) and u(t).

6.3 Results for the Time-Variant Problem without the Forcing Function

A FORTRAN program was written to solve the problem with the Galerkin method for the problem outlined above. The results for this are shown in Tables 6.1 and 6.2 , respectively.

Table 6.1 State Variable, x(t), for the time-varying problem

Time	Power Series [14]	5 Wavelets	17 Wavelets	65 Wavelets
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.912840	0.915685	0.913027	0.912859
0.2	0.844084	0.831370	0.843238	0.844035
0.3	0.792582	0.780250	0.791801	0.792536
0.4	0.757681	0.762327	0.757996	0.757706
0.5	0.739187	0.744403	0.739546	0.739215
0.6	0.737369	0.744091	0.737775	0.737398
0.7	0.752999	0.743779	0.752426	0.752966
0.8	0.787434	0.779642	0.786904	0.787406
0.9	0.842730	0.851680	0.843372	0.842774
1.0	0.921796	0.923717	0.921952	0.921811

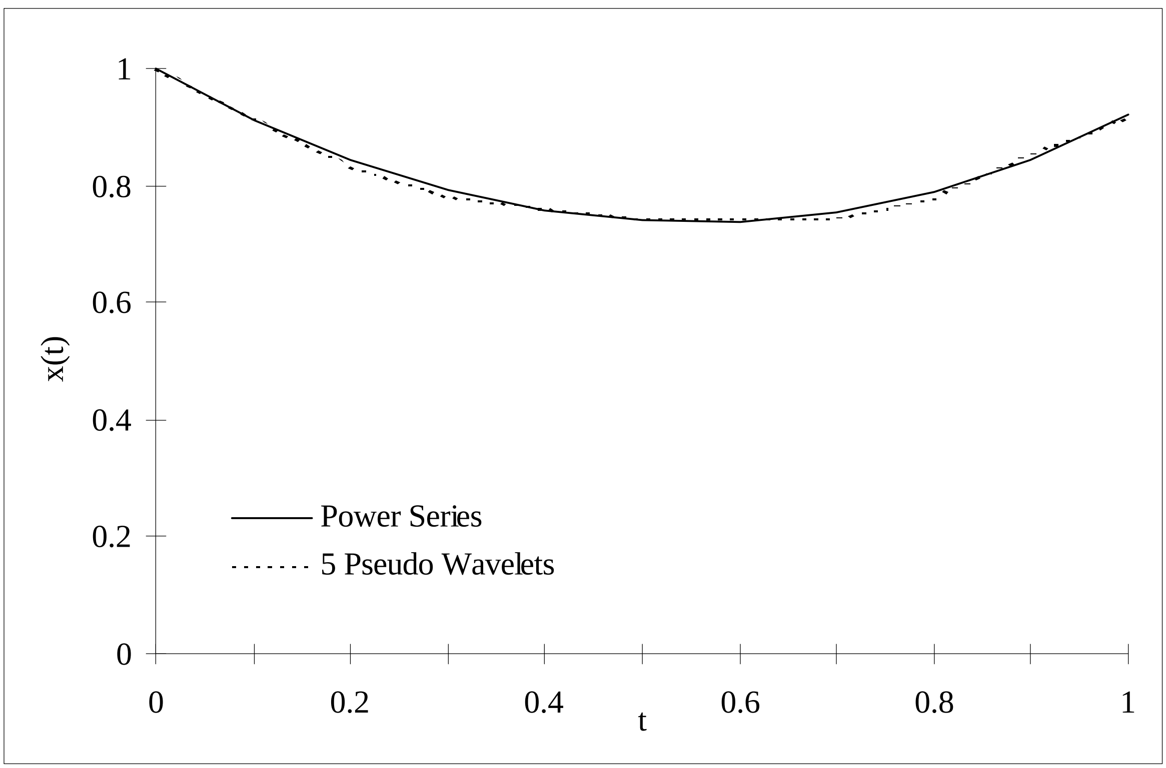

Figure 6.1 State Variable, x(t), for the time-varying problem

Table 6.2 Control Variable, u(t), for the time-varying problem

Time	Power Series	5 Wavelets	17 Wavelets	65 Wavelets
0.0	- 0.968348	- 0.958635	- 0.967868	-0.968493
0.1	- 0.868375	- 0.858714	- 0.867928	-0.868504
0.2	- 0.768502	- 0.758792	- 0.768018	-0.768608
0.3	- 0.668943	- 0.660378	- 0.668435	-0.669029
0.4	- 0.569985	- 0.563470	- 0.569591	-0.570060
0.5	- 0.471934	- 0.466562	- 0.471656	-0.471999
0.6	- 0.357122	- 0.368542	- 0.374731	-0.375099
0.7	- 0.279516	- 0.270522	- 0.278992	-0.279529
0.8	- 0.185333	- 0.177210	- 0.184778	-0.185324
0.9	- 0.092312	- 0.088605	- 0.091999	-0.092300
1.0	0.000000	0.000000	0.000000	0.000000

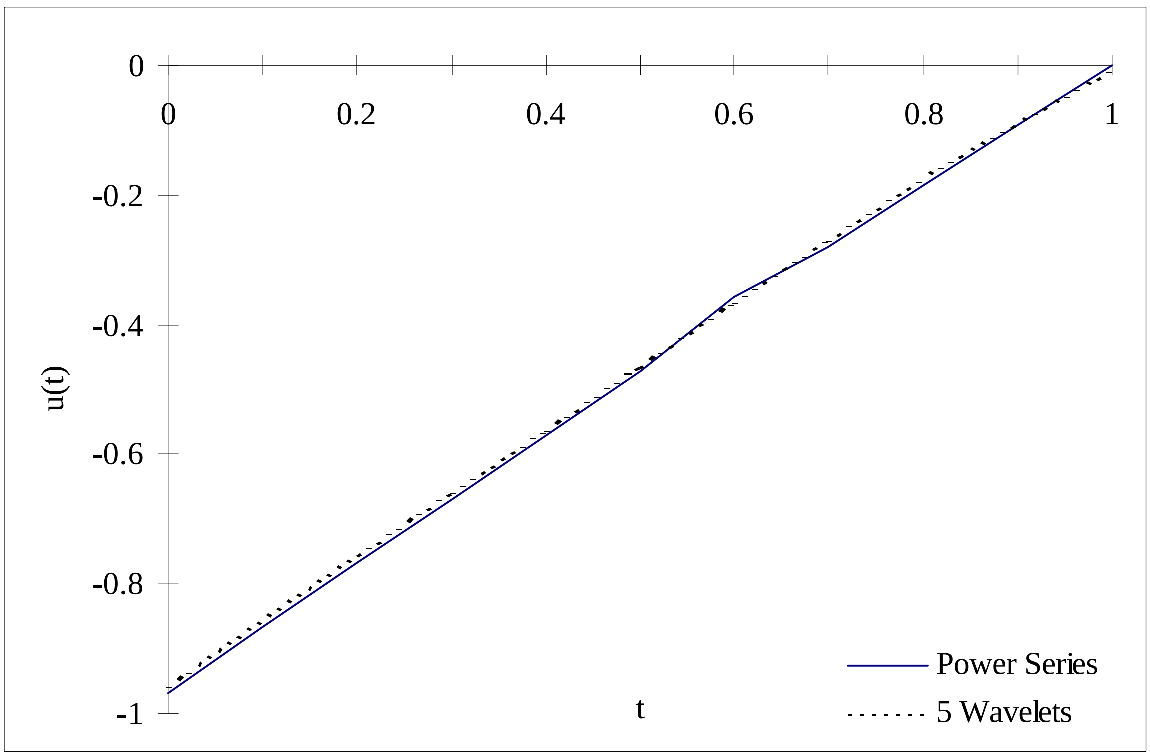

Figure 6.2 Control Variable, u(t), for the time-varying problem

6.4 Time Varying Problem with Step Forcing Function

After these results were obtained, a step function, F(t), was added to the dynamic constraints of the problem. This function can be representative of a switch which is closed at a certain time. The problem is very similar to the one just solved except (6.2a) is replaced by the following equation.

$$\dot{x}(t) = tx(t) + u(t) + F(t) \tag{6.6}$$

6.5 Results for Step Forcing Function

The forcing function applied in this section was a step function which was of height 0 on the interval [0,0.5] and 1 on the interval [0.5,1].

Table 6.3 State Variable, x(t), for the time-varying problem with step forcing function

Time	5 Wavelets	17 Wavelets	65 Wavelets
0.0	1.00000	1.00000	1.00000
0.1	0.89571	0.90078	0.90111
0.2	0.79143	0.81850	0.82029
0.3	0.72866	0.75469	0.75636
0.4	0.70740	0.70834	0.70837
0.5	0.68614	0.67632	0.67566
0.6	0.76652	0.76140	0.76116
0.7	0.84690	0.86923	0.87063
0.8	0.98839	1.00764	1.00899
0.9	1.19101	1.18293	1.18225
1.0	1.39363	1.39751	1.39772

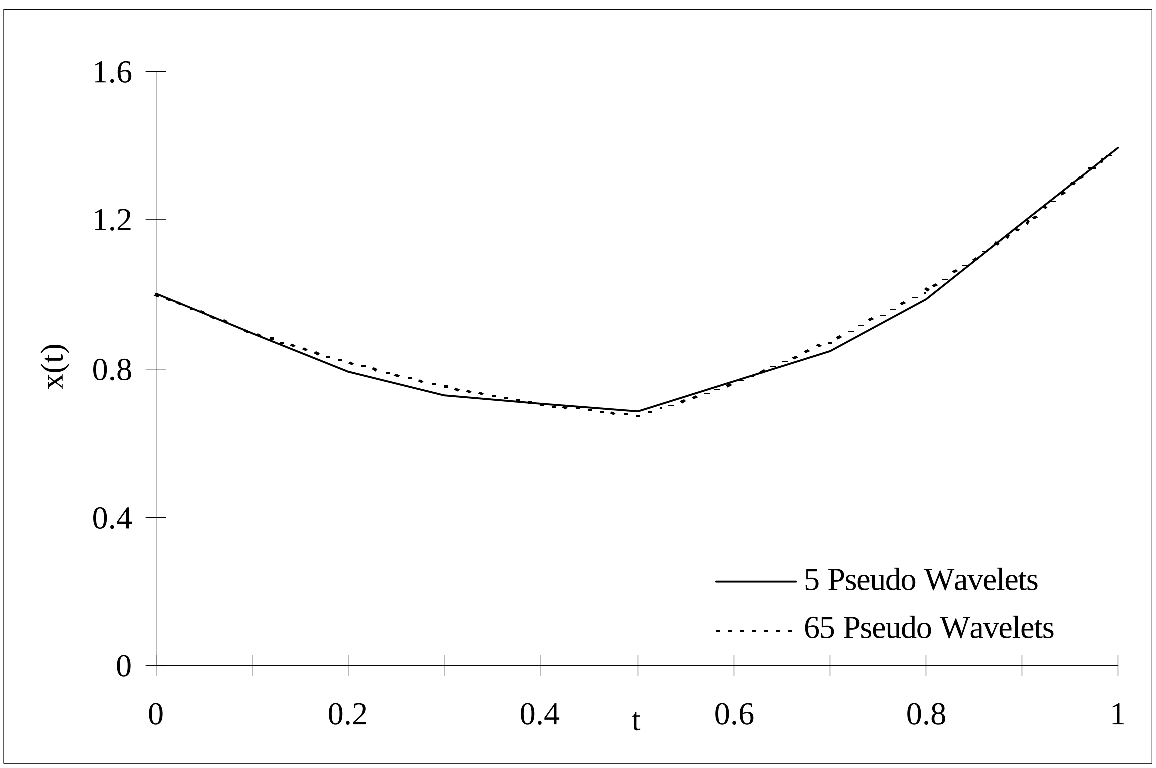

Figure 6.3 State Variable, x(t), for the time-varying problem with step forcing function

Table 6.4 Control Variable, u(t), for the time-varying problem with step forcing function

Time	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	-1.06046	-1.07879	-1.08367	-1.08522
0.1	-0.96135	-0.97902	-0.98383	-0.98524
0.2	-0.86223	-0.88006	-0.88397	-0.88536
0.3	-0.76549	-0.78078	-0.78450	-0.78585
0.4	-0.67111	-0.68191	-0.68597	-0.68706
0.5	-0.57673	-0.58608	-0.58858	-0.58937
0.6	-0.47033	-0.48411	-0.48709	-0.48811
0.7	-0.36393	-0.37491	-0.37775	-0.37862
0.8	-0.24858	-0.25736	-0.26012	-0.26095
0.9	-0.12429	-0.13179	-0.13417	-0.13493
1.0	0.00000	0.00000	0.00000	0.00000

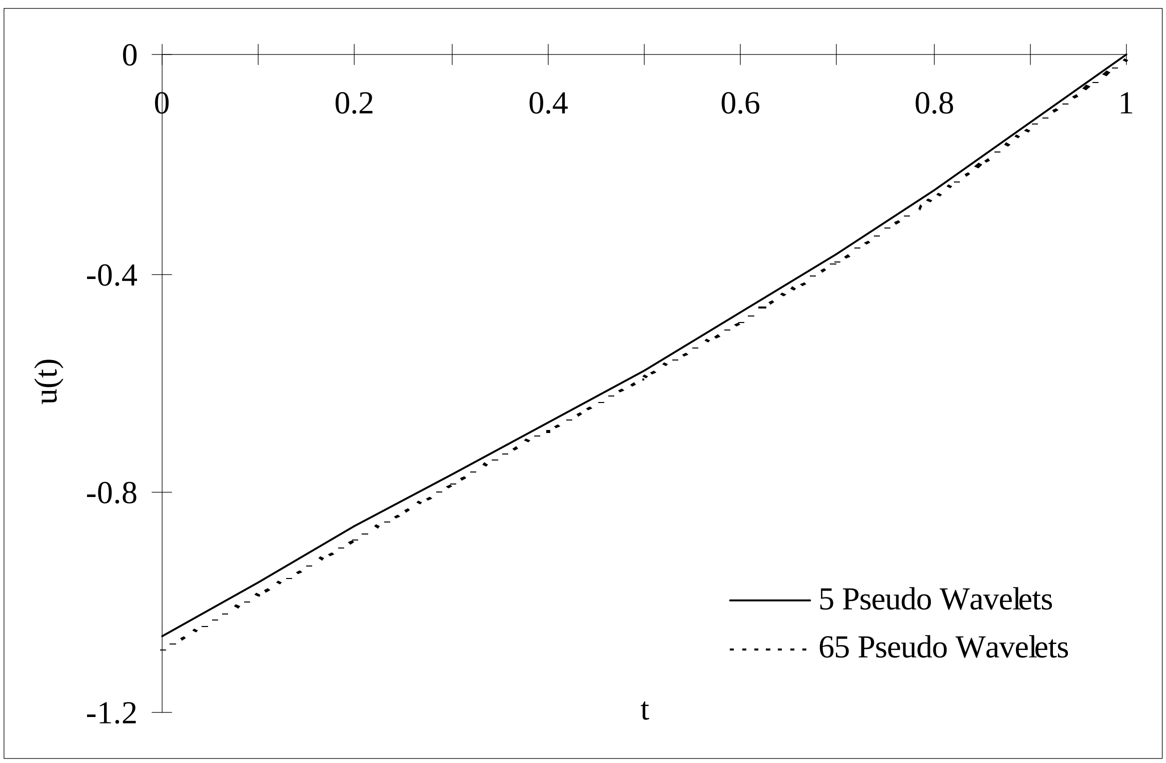

Figure 6.4 Control Variable, u(t), for the time-varying problem with step forcing function

6.6 The Time-Varying Problem with a Rectangular Pulse Forcing Function

A rectangular pulse was added to the dynamic constraints of the problem. This pulse was of amplitude 1 in the interval [0.5,0.75] and zero elsewhere. This could possibly represent some shock that the system was exposed to. This program could easily accommodate any choice of rectangular forcing function, and if one chooses a forcing function with large amplitude and small width, then this could be used to represent a delta function.

Table 6.5 State Variable, x(t), for the time-varying problem with rectangular pulse forcing function

Time	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	1.00000	1.00000	1.00000	1.00000
0.1	0.90665	0.898300	0.90078	0.90403
0.2	0.81330	0.82664	0.81850	0.82619
0.3	0.75157	0.76569	0.75469	0.76534
0.4	0.72145	0.71584	0.70834	0.72060
0.5	0.69132	0.69347	0.67632	0.69140
0.6	0.78398	0.77616	0.76140	0.78081
0.7	0.87663	0.89665	0.86923	0.89464
0.8	0.96714	0.98901	1.00764	0.98696
0.9	1.05550	1.05322	1.18293	1.05636
1.0	1.14386	1.15526	1.39751	1.15543

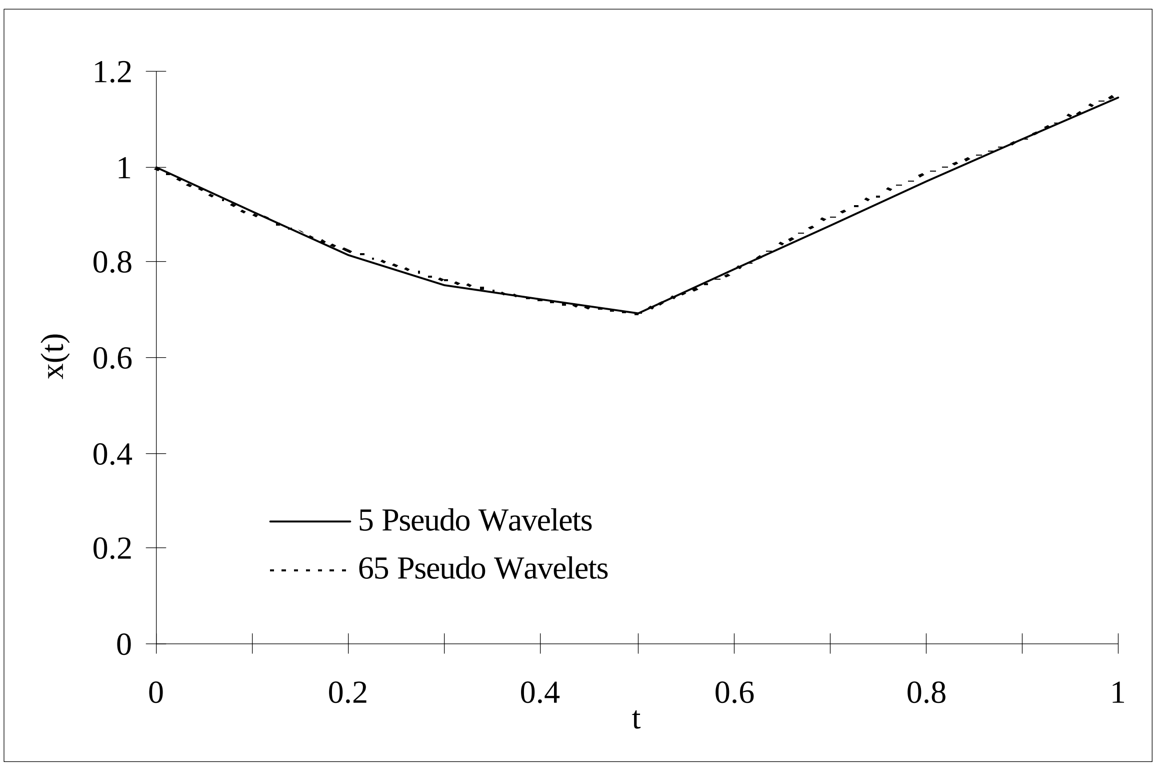

Figure 6.5 State Variable, x(t), for the time-varying problem with rectangular pulse forcing function

Table 6.6 Control Variable, u(t), for the time-varying problem with rectangular pulse forcing function

Time	5 Wavelets	9 Wavelets	17 Wavelets	65 Wavelets
0.0	-1.05791	-1.05191	-1.05526	-1.05632
0.1	-0.95791	-0.95202	-0.95537	-0.95634
0.2	-0.85792	-0.85280	-0.85549	-0.85646
0.3	-0.75942	-0.75339	-0.75597	-0.75693
0.4	-0.66241	-0.65443	-0.65733	-0.65809
0.5	-0.56541	-0.55799	-0.55975	-0.56031
0.6	-0.45984	-0.45619	-0.45811	-0.45889
0.7	-0.35428	-0.34624	-0.34854	-0.34913
0.8	-0.24120	-0.23028	-0.23151	-0.23229
0.9	-0.12060	-0.11267	-0.11527	-0.11569
1.0	0.00000	0.00000	0.000000	0.00000

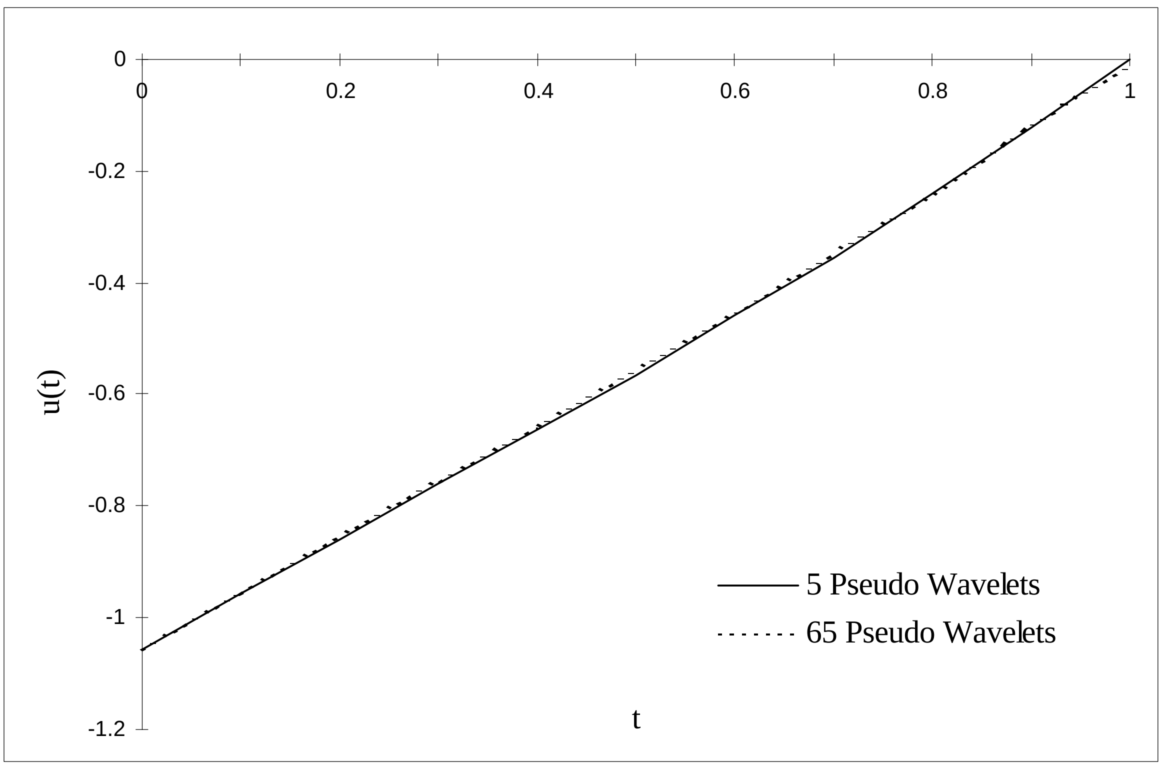

Figure 6.6 Control Variable, u(t), for the time-varying problem with rectangular pulse forcing function

6.7 Conclusion

In this chapter the time-variant problem was considered. Many of the methods were similar to those used for the time invariant problem and many of the matrices were the same. This supports the fact that this wavelet Galerkin method is very flexible and can be used to solve a large variety of optimal control problems.

Chapter 7: The Time-Delay Problem

7.1 Introduction

The time-delay problem is similar in many aspects to the problems that have been previously solved. The main difference is that there is a time delay ,τ , included in the constraint equation. The general formulation introduced in chapter four does not apply to this problem.

7.2 Statement a General Time-Delay Problem

A general time - delay problem can be stated as follows [10].

minimize $$J = \frac{1}{2}\int_0^{t_f} u(t)^T R u(t) dt \qquad (7.1)$$

subject to the following constraints

$$x(t) = \alpha(t) \qquad t \varepsilon [-\tau, 0] \qquad (7.2)$$

$$x(t_f) = x_f \qquad (7.3)$$

Where x(t) is the state vector and u(t) is the control vector. A, B, and C are all matrices that may be time dependent. $\alpha(t)$ is a known function and τ is a known constant that represents the time delay.

7.3 General Solution Method

It should be noted that the if it were not for the constraint $x(t_f) = x_f$, then the control variable, u(t) would be identically zero. This will cause computational problems with many algorithms if the problem is not reformulated. The problem can be reformulated as follows.

$$J = \frac{1}{2}\xi(x(t_f) - x_f)^2 + \frac{1}{2}\int_0^{t_f} u(t)^T R u(t) dt$$
$$= \frac{1}{2}\int_0^{t_f} u(t)^T R u(t) + \xi(x(t_f) - x_f)^2 \, \delta(t) dt \qquad (7.4)$$

subject to the following constraints

$$x(t) = \alpha(t) \qquad t \, \varepsilon [-\tau, 0] \qquad (7.5)$$

$$x(t_f) = x_f \qquad (7.6)$$

Here ζ is some positive number. In the limit as $\zeta \to \infty$, this reformation becomes the same as the problem introduced in the beginning of the chapter. Using a langrange multiplier technique for the calculus of variations, the following equation is obtained.

$$\int_0^{t_f} [\delta x^T (\dot{x}(t) - Ax(t) - Bx(t-\tau) - CR^{-1}C^T p(t))] dt = 0 \qquad (7.7)$$

$$-2\zeta(x(t_f) - x_f) + \int_0^{t_f} [\delta p^T [A^T p(t)) + B^T p(t+\tau)]] dt = 0 \qquad (7.8)$$

When these differential equations are expanded in terms of the pseudo wavelets, a linear system of equations is obtained and can be solved by matrix algebra.

7.4 A Sample Time-Delay Problem

Consider the following time-delay system.

$$\text{minimize } J = \frac{1}{2}\int_0^{t_f} u(t)^2 dt \tag{7.9}$$

This is subject to the following constraints.

$$\dot{x}(t) = x(t-1) + u(t) \tag{7.10}$$

$$x(t) = 1 \qquad t \varepsilon [-1,0] \tag{7.11}$$

$$x(2) = 0 \tag{7.12}$$

This problem leads to the following equations.

$$\int_0^2 [\delta x^T(\dot{x}(t) + x(t-1) - p(t))]dt = 0 \tag{7.13}$$

$$-2\zeta x^2(2) + \int_0^2 [\delta p^T[p(t)) - p(t+1)]]dt = 0 \tag{7.14}$$

Before x(t) and p(t) are expanded, the fact that the value of x(t) in [-1,0] is 1 and the fact that p(t) in [2,3] is 0 must be applied. After this, is assumed that both the state variable x(t) and the control variable u(t) can be expanded in terms of the triangle functions, $\phi_n(t)$, discussed in chapter three. So the assumption made is:

$$u(t) = \sum_{k=1}^{N} b_k f_k \tag{7.15}$$

$$x(t) = \sum_{k=1}^{N} a_k f_k(t) \tag{7.16}$$

Note that since the initial value x is known and completely determined by the function ϕ_2 and the final value of x is known and is completely determined by ϕ_1 . From this information, the coefficients a_2 = 1 and a_2 = 0. Using equations (7.15) and (7.16) with the included definitions leads one to the following result:

$$\left[\int_0^2 \Phi_{(1)}(t)\dot{\Phi}_{(1)}^T(t)dt + \int_1^2 \Phi_{(1)}(t)\Phi_{(1)}^T(t-1)dt\right] * C - \int_0^1 \Phi_{(1)}\Phi_{(2)}^T dt * D$$

$$= \frac{1}{2}\int_1^2 \Phi_{(1)}(t)\phi_{(2)}(t-1)dt + \int_0^1 \Phi_{(1)}dt \qquad (7.17)$$

$$-8\zeta a_1 + \left[\int_0^2 \Phi_{(2)}(t)\dot{\Phi}_{(2)}(t)dt * C - \int_1^2 \Phi_{(2)}(t)\Phi_{(2)}^T(t+1)dt\right] * D = 0 \qquad (7.18)$$

Now if we let X=[C D] and

$$Z = \begin{bmatrix} \int_0^2 \Phi_{(1)}(t)\dot{\Phi}_{(1)}^T(t)dt + \int_1^2 \Phi_{(1)}(t)\Phi_{(1)}^T(t-1)dt & -\int_0^1 \Phi_{(1)}\Phi_{(2)}^T dt \\ -8\zeta\begin{bmatrix} 1\ 0\ 0\ 0\ldots.0 \\ 1\ 0\ 0\ 0\ldots.0 \\ . \\ . \\ 1\ 0\ 0\ 0\ldots.0 \end{bmatrix} & \int_0^2 \Phi_{(2)}(t)\dot{\Phi}_{(2)}(t)dt - \int_1^2 \Phi_{(2)}(t)\Phi_{(2)}^T(t+1)dt \end{bmatrix}$$

$$Y = \begin{bmatrix} \frac{1}{2}\int_1^2 \Phi_{(1)}(t)\phi_{(2)}(t-1)dt + \int_0^1 \Phi_{(1)}dt \\ 0 \end{bmatrix}, \qquad (7.19)$$

then this system can be expressed in the form ZX = Y. These are a set of 2n-2 equations and 2n-2 unknowns. This is a sparse system in the sense that many of the elements in the problem matrix are zero.

This system of equations gives the need to compute 2 matrices, $\int_1^2 \Phi_{(1)}(t)\Phi_{(1)}^T(t-1)dt$ and $\int_1^2 \Phi_{(2)}(t)\Phi_{(2)}^T(t+1)dt$.

Since the triangular functions are being used, Simpson's Rule for integration turns out to be exact as long as the grid points are chosen correctly. Very few grid points are necessary since the elements of the integrand are a second degree curve between corners.

7.5 Results for the Sample Time Delay Problem

The value of the parameter ζ had little bearing on the solution. As long as this was greater than 0, the zero solution for u(t) was avoided. Within 8 decimal places, the values of the state and control variables were the same as long as $\zeta > 10^{-12}$.

Table 7.1 State Variable, x(t), for the time-delay problem

t	Exact	Liou and Chou[10] (n=8)	3 Pseudo Wavelets	5 Pseudo Wavelets	9 Pseudo Wavelets	17 Pseudo Wavelets
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.25	0.7617	0.7618	0.7188	0.7734	0.7617	0.7616
0.50	0.5469	0.5471	0.4375	0.5469	0.5469	0.5469
0.75	0.3555	0.3559	0.1563	0.3672	0.3555	0.3555
1.00	0.1875	0.1882	-0.1250	0.1875	0.1875	0.1875
1.25	0.0615	0.0621	-0.0938	0.0768	0.0550	0.0615
1.50	-0.0078	-0.0073	-0.0625	-0.0339	-0.0078	-0.0078
1.75	-0.0264	-0.0261	-0.0313	-0.0169	-0.0329	-0.0264
2.00	0.0000	0.0000	-0.0000	0.0000	-0.0000	0.0000

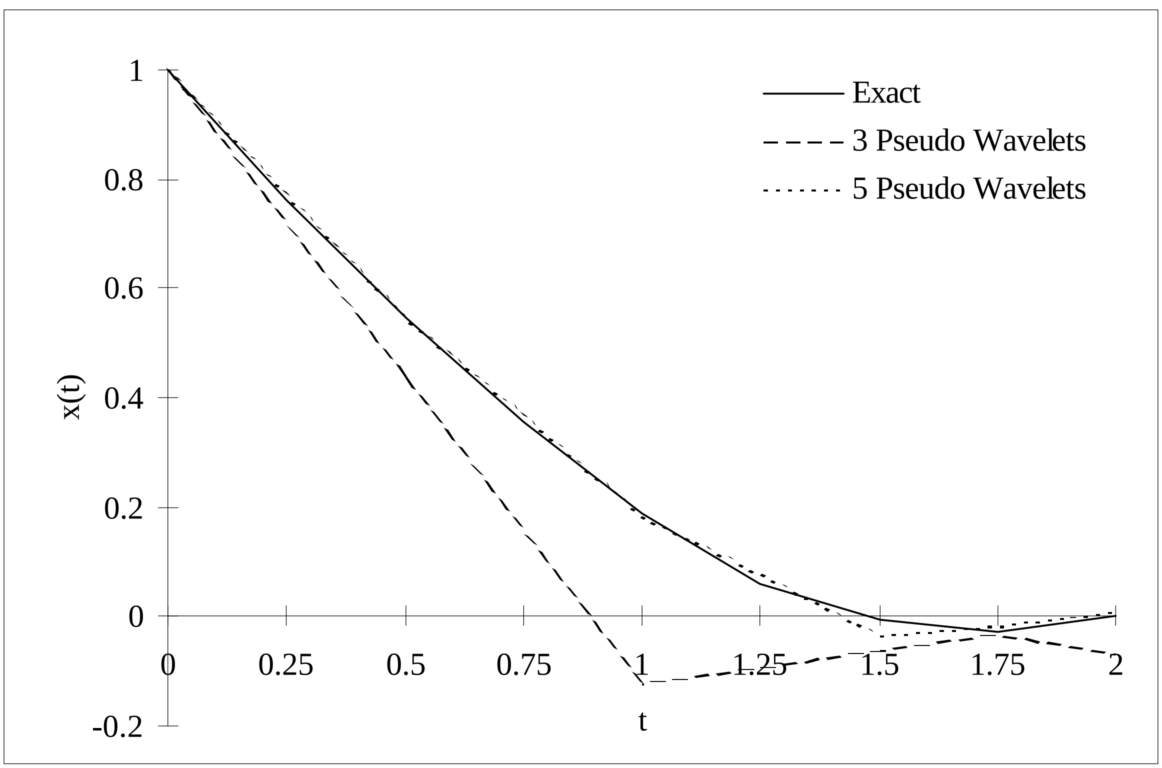

Figure 7.1 State Variable, x(t), for the time-delay problem

Table 7.2 Control Variable, u(t), for the time-delay problem

Time	Exact Solution	Liou and Chou[10] (n=8)	3 Pseudo Wavelets	5 Pseudo Wavelets	9 Pseudo Wavelets	17 Pseudo Wavelets
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.0938	0.0941	0.0938	0.0938	0.0938	0.0938
0.50	0.1875	0.1882	0.1875	0.1875	0.1875	0.1875
0.75	0.2813	0.2824	0.2813	0.2813	0.2813	0.2813
1.00	0.3750	0.3764	0.3750	0.3750	0.3750	0.3750
1.25	0.3750	0.3764	0.3750	0.3750	0.3750	0.3750
1.50	0.3750	0.3764	0.3750	0.3750	0.3750	0.3750
1.75	0.3750	0.3764	0.3750	0.3750	0.3750	0.3750
2.00	0.3750	0.3764	0.3750	0.3750	0.3750	0.3750

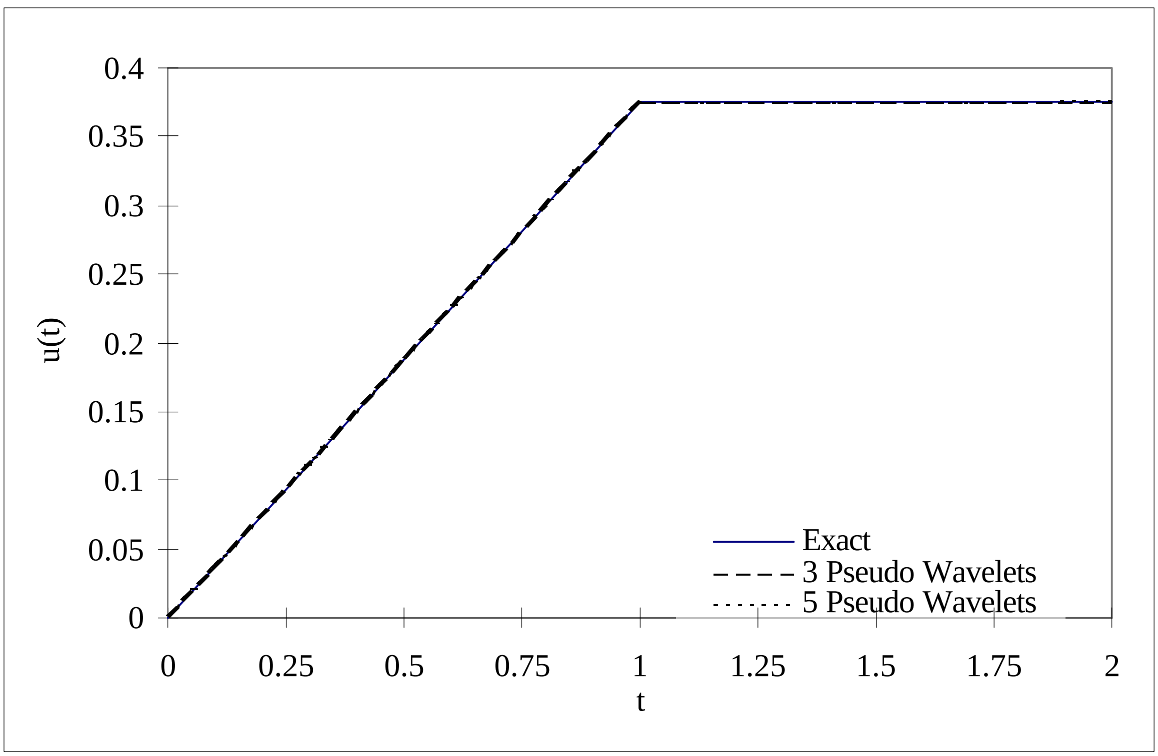

Figure 7.2 Control Variable, u(t), for the time-delay problem

7.6 Conclusion

In this chapter, the application of pseudo wavelets to the optimal control problem has been discussed. These results are more accurate than the numerical results published by Liou and Chou in 1987. Their algorithm used a recursive method to solve the problem. The final value of the control variable affects the solution in this problem, and this value was captured by introducing the final value of the state variable into the performance index of the problem. These results are very accurate and the value of the control variable is captured exactly in only three basis functions. Note that the nature of this problem and most time-delay problems of this sort may lead to solutions which are not differentiable at some point. Triangular pseudo wavelets are well suited to solving problems of this nature.

Chapter 8: Conclusion

8.1 Chapter Summary

This thesis has discussed ways in which wavelets and pseudo-wavelets can be applied to numerical methods. In chapter one, the basic properties of wavelets and the generation of Daubechies Wavelets were discussed. These wavelets are constructed in such a way that they will have convenient properties. The Haar wavelets are a type of Daubechies wavelets with two coefficients in the dilation equation. In chapter two, the application of the Haar basis to the MEI (Measured Equation of Invariance) method was discussed. Computational times were found to be about the same as for the sine basis. Although it was not done here, many fast algorithms exist which can make the Haar basis perform considerably faster. In Chapter three, the pseudo wavelets were discussed. These pseudo wavelets have the property that they capture the initial and final value of the function with a single function. They also have the property that their derivative is the Haar basis. In chapter four, a very general formulation was discussed in for which the pseudo wavelets could be used to obtain a linear system of equations, which can be solved by any linear solver. In chapter five, the time-invariant problem was discussed. It was found that the wavelets converged to solution in relatively few basis functions. Also, since two of the wavelets are determined by the initial and final values, the number of unknowns in the linear system was reduced. In Chapter six, it was shown that these pseudo wavelets applied to the Galerkin method also provide satisfactory results. In chapter seven, the pseudo wavelets were applied to the time delay problem. It was found that the solution was obtained in very few approximating functions and the final value of the control variable was captured.

8.2 Concluding Remarks

In conclusion, wavelet concepts have been applied to numerical problems to demonstrate how they can improve computational time and efficiency. For all the results in this thesis, it was found that the results with the wavelets and triangle functions were comparable, or an improvement on the results found in previously published results. In optimal control, it was found that the pseudo-wavelets used in Galerkin solution led to a linear system of equations. The choice of triangle functions as a basis reduced the number of unknowns to be solved for by the number of initial and final conditions given in the problem. This will lead to a decrease in computational time, since the majority of the time involved in these problems is solving the system of equations.

8.3 Future Directions

There are many algorithms associated with wavelets and techniques that can be used to increase computational time. One such technique is the idea of multiresolution. In multiresolution, algorithms are used that take advantage of the localized behavior. If there is a certain area where there is much changing, more wavelet functions can be used in that area and less elsewhere. This is one advantage of wavelets and pseudo-wavelets over entire domain basis functions. None of these fast methods were applied in this paper, but they could have been to obtain faster computational times.

In optimal control constraints involving inequalities could be considered. These types of applications are well suited to wavelets and triangle functions since the constraint may switch between being active and inactive. Furthermore, this paper has only considered linear problems. Nonlinear problems could be researched, but this would lead to some nonlinear system of algebraic equations that could not be solved with a standard equation solver.

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